Compact difference schemes for rod lateral vibrations equation

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Introduction

We consider finite-difference approximations for the rod lateral vibrations equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x}\left[R^2\rho\frac{\partial^3 u}{\partial x\partial t^2}\right]+\frac{\partial^2}{\partial x^2}\left[ER^2\frac{\partial^2 u}{\partial x^2}\right]=f\,, \tag{1}$$

where ρ is rod's density, *R* – radius, *E* – Young module; $x\in[0,\mathrm{L}]$, *f=f(t, x)* - forcing. The equation is not resolved with respect to higher temporal derivative, i. e. it has not Cauchy – Kovalevsky type, but Poincare – Sobolev one. However, it is not an obstacle for its high-order approximation.

We have investigated both cases: R = *const*, and *R = R(x)*. We compare high-order difference compact and Crank - Nicolson schemes. We compare the following properties: order of approximation, stability, energy conservation law (for homogeneous case *f=0*). The case of the variable coefficients of the differential equation is much more difficult for a good approximation. High-order compact approximation for a set of boundary conditions is also discussed.

Compact difference scheme

We use the following 3-5-3-point stencil for a compact difference scheme, which can be expressed as a linear algebraic equation in values of grid functions *u* and *f* in stencil's points.

$$\begin{aligned}
&a^*(u_0^{n+1}+u_0^{n-1})+a_{left}(u_{-h}^{n+1}+u_{-h}^{n-1})+a_{right}(u_h^{n+1}+u_h^{n-1})+bu_0^n+\\
&+c_{left}u_{-h}^n+c_{right}u_h^n+d_{left}u_{-2h}^n+d_{right}u_{2h}^n=p_{0,\mathrm{left}}(f_{-2h,j}^{n+1}+f_{-2h,j}^{n-1})+p_{0,right}(f_{2h,j}^{n+1}+f_{2h,j}^{n-1})+\\
&+q_{0,left}(f_{-h,j}^{n+1}+f_{-h,j}^{n-1})+q_{0,right}(f_{h,j}^{n+1}+f_{h,j}^{n-1})+r_0(f_{0,j}^{n+1}+f_{0,j}^{n-1})+\\
&+p_{1,left}f_{-2h,j}^n+p_{1,right}f_{2h,j}^n+q_{1,left}f_{-h,j}^n+q_{1,right}f_{h,j}^n+r_1f_{0,j}^n\,.
\end{aligned} \tag{2}$$

Here *n* is a temporal step number, and the lower index shows spatial position of stencil points over the stencil's center. The seventeen constants are calculated for every spatial grid point x_j by substituting the following test monomials $u_k(t,x)$ (see Fig. 1) and the corresponding right-hand sides, calculated according to (1); the similar approach see [Gordin, Tsymbalov, 2014]:

$$f_{kj}(t,x)=\frac{\partial^2 u_k}{\partial t^2}-R_j^2\frac{\partial^4 u_k}{\partial x^2\partial t^2}-2R_j(\mathrm{R}_j')\frac{\partial^3 u_k}{\partial x\partial t^2}+2E\rho^{-1}(\mathrm{R}_j')^2\frac{\partial^2 u_k}{\partial x^2}+$$

$$+ER_j^2\rho^{-1}\frac{\partial^4 u_k}{\partial x^4}+4ER_j(\mathrm{R}_j')\frac{\partial^3 u_k}{\partial x^3}+2ER_j\rho^{-1}\frac{\partial^2 u_k}{\partial x^2}(\mathrm{R}_j'').$$

Here $\mathrm{R}_j, \mathrm{R}_j', \mathrm{R}_j''$ are values of $R(x), \frac{\partial R(x)}{\partial x}, \frac{\partial^2 R(x)}{\partial x^2}$ at x_j. Derivatives could be evaluated either analytically or numerically (using high-order compact relations for the first and second derivatives, see (Patterson, 1983)), according to the problem.

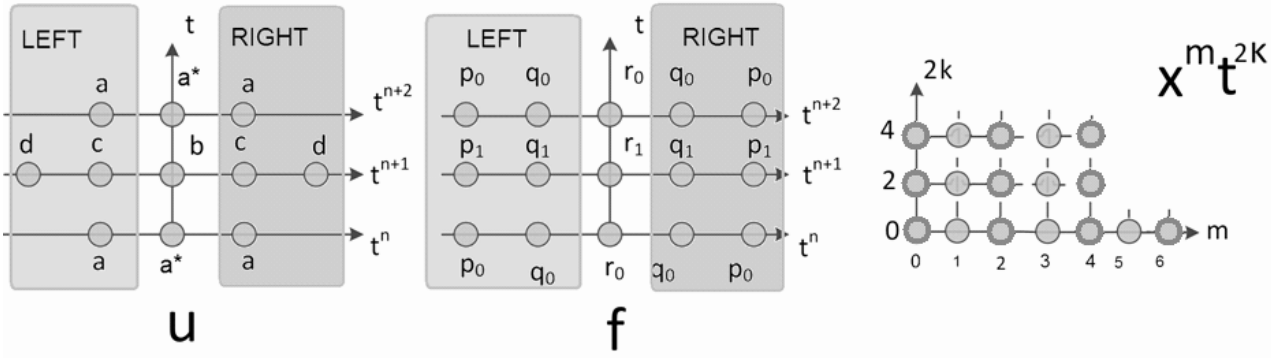

Fig. 1. Stencil and Newton's diagram of test monomials $u_k(t,x)$ for compact difference scheme (2).

Crank - Nicolson scheme

Stencil for Crank-Nicolson-type scheme can be written as:

$$a^*(u_0^{n+1}+u_0^{n-1})+a_{left}(u_{-h}^{n+1}+u_{-h}^{n-1})+a_{right}(u_h^{n+1}+u_h^{n-1})+e_{left}(u_{-2h}^{n+1}+u_{-2h}^{n-1})+$$
$$+e_{right}(u_{2h}^{n+1}+u_{2h}^{n-1})+bu_0^n+c_{left}u_{-h}^n+c_{right}u_h^n=p_{0,\mathrm{left}}(f_{-2h,j}^{n+1}+f_{-2h,j}^{n-1})+$$
$$+p_{0,right}(f_{2h,j}^{n+1}+f_{2h,j}^{n-1})+q_{0,left}(f_{-h,j}^{n+1}+f_{-h,j}^{n-1})+q_{0,right}(f_{h,j}^{n+1}+f_{h,j}^{n-1})+$$
$$+r_0(f_{0,j}^{n+1}+f_{0,j}^{n-1})+p_{1,left}f_{-2h,j}^n+p_{1,right}f_{2h,j}^n+q_{1,left}f_{-h,j}^n+q_{1,right}f_{h,j}^n+r_1f_{0,j}^n. \qquad (3)$$

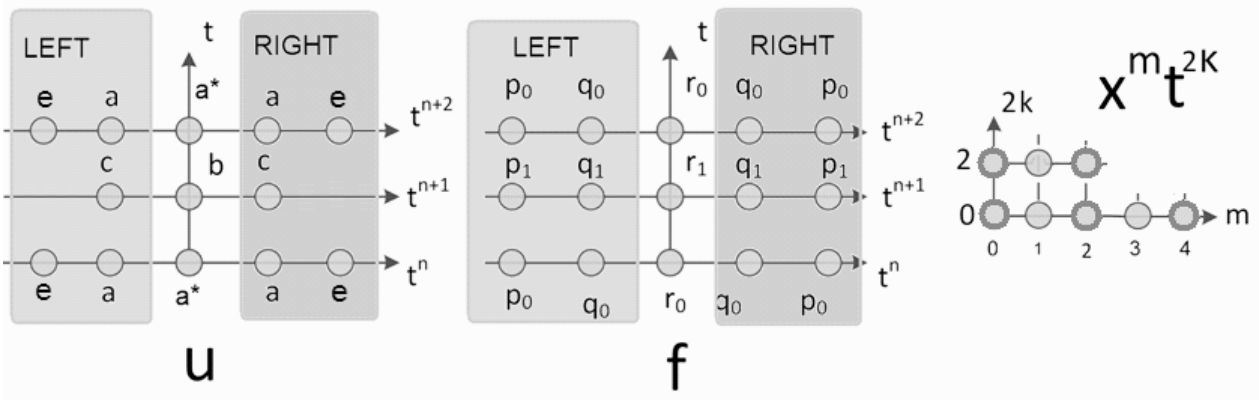

Fig. 2. Stencil and Newton's diagram of test monomials $u_k(t,x)$ for Crank-Nicolson-type scheme (3).

Stability

Our numerical experiments demonstrated:

- The conditional stability of the compact difference scheme (2) if $\nu^* < \mu^* + 1/12$;
  here and after $\mu^* = (R^*)^2 h^{-2}$, $\nu^* = (R^*\tau)^2 E\rho^{-1}h^{-4}$, $R^* = \max_j R(x_j)$.
- The absolute stability of Crank-Nicolson scheme (3).

We have also investigated stability issues for a small number of spatial grid points *N* in case of low smoothness order of rod's radius function *R(x)*.

Numerical experiments

For schemes' errors evaluation, we use mesh norms C and L_2 as well as the mesh energy norm

$$\sqrt{\frac{R(x)^2\left\|\left(\partial_t u_{anal} - \delta_t u_{diff}\right)^2 + ER(x)^2\rho^{-1}\left(\partial_x u_{anal} - \delta_x u_{diff}\right)^2\right\|_{L_2}}{\left\|R(x)^2\left(\partial_t u_{anal}\right)^2 + ER(x)^2\rho^{-1}\left(\partial_x u_{anal}\right)^2\right\|_{L_2}}}. \qquad (4)$$

Here u_{anal} is analytical solution, u_{diff} is a difference one. Usage of norm (3) allows us to account the kinetic part of solution, while standard C and L_2 mesh norms ignore it.

Table 1. Errors and orders of accuracy of the solution of (1) with compact difference scheme (2) (left) and Crank - Nicolson-type scheme (3) (right). Orders of accuracy exceed fourth for both schemes in energy norm, second order for norms C and L_2. Scheme (2) is more accurate than (3). $L = 4\pi, \rho = 7000, E = 2.1*10^8, u_{test} = \sin(x)\sin(t) + 2, R(x) = 0.4 + 0.01\cos^2(x), T = 0.2, \nu^* = 0.05$

Norm	N = 12	N = 24	N = 48	N = 96	RMS
C	1.99-3	6.72-4	1.75-4	4.42-5	1.84
L_2	1.62-3	4.75-4	1.24-4	3.13-5	1.90
(4)	3.59-5	9.67-7	4.41-8	2.52-9	4.62

	N = 12	N = 24	N = 48	N = 96	RMS
C	3.70-2	9.30-3	2.24-3	5.56-4	2.02
L_2	3.02-2	6.57-3	1.59-3	3.93-4	2.09
(4)	3.91-2	1.80-3	1.04-4	6.38-6	4.20

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