Volatility forecasting using global stochastic financial trends extracted from asynchronous market quotes

Lyudmila Grigoryeva[1], Juan-Pablo Ortega[2], and Anatoly Peresetsky[3]

[1]Universität Konstanz, Germany

[2]CNRS, France

[3]Higher School of Economics, Moscow, Russia

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Outline

1. The global stochastic trend and asynchronous intraday returns
2. Estimation via Kalman models
3. Volatility modeling/forecasting and need for nonlinear models
4. Forecasting setup
5. Empirical study and results

A motivation: the Capital Asset Pricing Model (CAPM)

Capital Asset Pricing Model (CAPM) related regression

$$r_t = \alpha + \beta y_t + u_t \quad \text{with} \quad \{u_t\} \sim \mathrm{WN}(0, \sigma^2),$$

where the presence of risk-free assets is neglected.

Standard approaches to the modeling of y_t:

- Use a proxy index constructed out of a set of assets that represent the class to which r_t is naturally associated.
- Treat the market returns as a non-observable variable and to extract it using observed individual returns via a Kalman type state-space model: low sampling frequencies in order to be able to neglect asynchronicity issues; daily quoted data are used but always using a synchronized approach.

The idea: asynchronous intraday returns and the GST

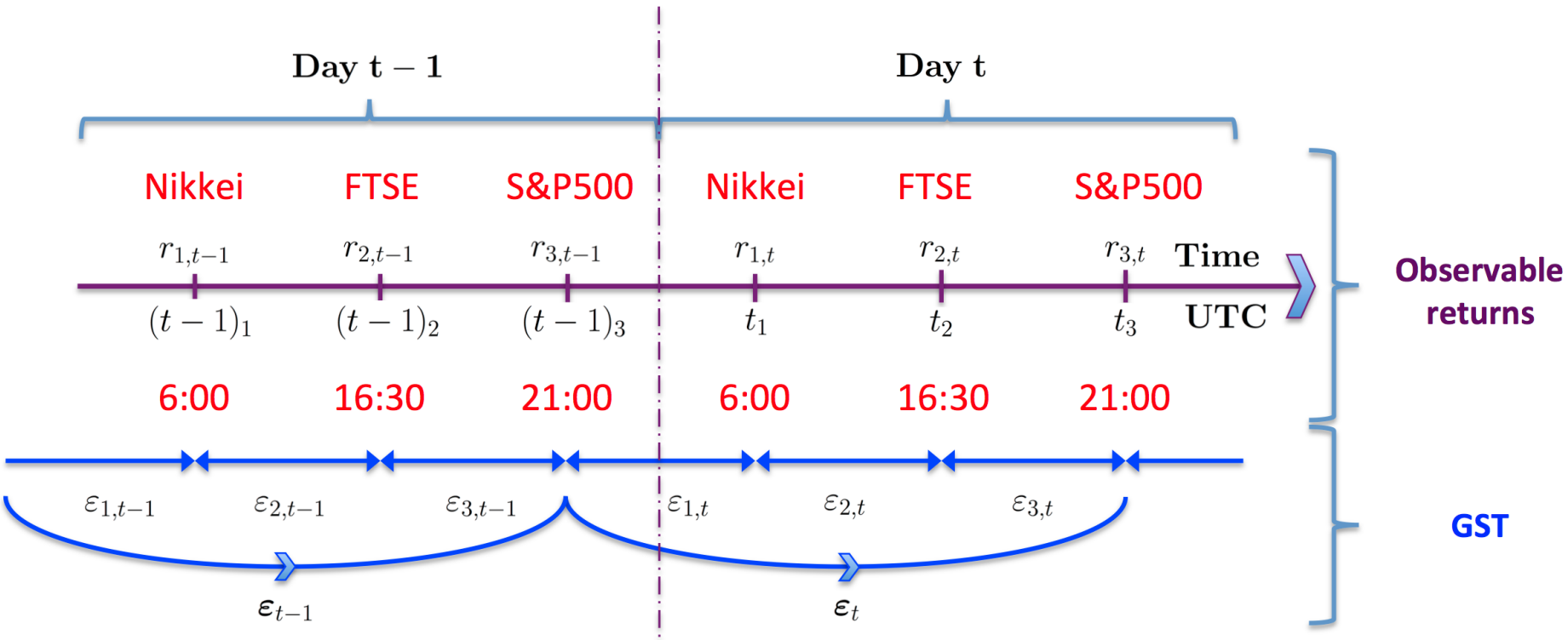


Figure 1 : The chronology of the intraday returns and the global stochastic trend.

Proposed approach:

- Market returns are thought of as a non-observable global stochastic trend (GST) whose value is ruled by the arrival of information coming from different local markets.
- The returns of the GST are estimated several times per day at time points that are synchronized with the closing times of the markets that are assumed to drive it.
- Approach is implemented by setting a state-space model in which the observation equation writes the different observable individual market returns that we are interested in as a stochastic perturbation of an affine function of the estimated GST return accumulated during the 24 hours that precede this quote.
- It is assumed that the observed returns are those that drive the GST and hence the GST returns are estimated as many times per day as different closing times are included in the list of markets considered.

[GOP15] LG, J.-P. Ortega, and A. Peresetsky. 2015. Volatility forecasting using global stochastic financial trends extracted from non-synchronous data.

GST: state variables and CAPM scheme

Assume an underlying and non-observable global stochastic trend (GST).

Construct the **state variables** $\varepsilon_t \in \mathbb{R}^3$ for each $t \in \mathbb{N}$ as

$$\boldsymbol{\varepsilon}_t = \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{pmatrix} := \begin{pmatrix} s_{t_1} - s_{(t-1)_3} \\ s_{t_2} - s_{t_1} \\ s_{t_3} - s_{t_2} \end{pmatrix}, \tag{1}$$

where s_{t_i}, $t_i := 3(t-1)+i$, $i \in \{1,2,3\}$ are the intra t-day log-values of GST.

Use the CAPM scheme:

$$r_{i,t} = \alpha_i + \beta_i y_{i,t} + u_{i,t}, \quad i = \{1,2,3\}, \quad t \in \mathbb{Z}, \tag{2}$$

where $\alpha_i \in \mathbb{R}$, $\boldsymbol{\beta} := (\beta_1, \beta_2, \beta_3)^\top \in \mathbb{R}^3$, the vector $\mathbf{y}_t := (s_{t_1} - s_{(t-1)_1}, s_{t_2} - s_{(t-1)_2}, s_{t_3} - s_{(t-1)_3})^\top$ contains the daily GST returns synchronous with the components of $\mathbf{r}_t$.

The CAPM assumes $\{\mathbf{u}_t\} \sim \mathrm{WN}(\mathbf{0}_3, \Sigma_u)$, $\Sigma_u := \mathrm{diag}\left(\sigma^2_{u,1}, \sigma^2_{u,2}, \sigma^2_{u,3}\right)$ with $\sigma_{u,1}, \sigma_{u,2}, \sigma_{u,3} \in \mathbb{R}^+$.

Linear state-space model

Consider the linear state-space model for the GST [KP13]:

$$\begin{cases} \mathbf{r}_t = \boldsymbol{\alpha} + B\mathbf{e}_t + \mathbf{u}_t, & \{\mathbf{u}_t\} \sim \mathrm{WN}\left(\mathbf{0}_3, \Sigma_u\right), \quad (3a) \\ \mathbf{e}_t = T\mathbf{e}_{t-1} + R\mathbf{v}_{t-1}, & \{\mathbf{v}_t\} \sim \mathrm{WN}(\mathbf{0}_3, \mathbb{I}_3), \quad (3b) \end{cases}$$

where $\mathbf{e}_t := \left(\varepsilon_{1,t}, \varepsilon_{2,t}, \varepsilon_{3,t}, \varepsilon_{2,t-1}, \varepsilon_{3,t-1}\right)^\top$, $\boldsymbol{\alpha} \in \mathbb{R}^3$, $\Sigma_u := \mathrm{diag}\left(\sigma^2_{u,1}, \sigma^2_{u,2}, \sigma^2_{u,3}\right)$, $\sigma_{u,1}, \sigma_{u,2}, \sigma_{u,3} \in \mathbb{R}^+$, the matrices $B \in \mathbb{M}_{5,3}$, $T \in \mathbb{M}_5$, and $R \in \mathbb{M}_{3,5}$ are defined as

$$B := \begin{pmatrix} \beta_1 & 0 & 0 & \beta_1 & \beta_1 \\ \beta_2 & \beta_2 & 0 & 0 & \beta_2 \\ \beta_3 & \beta_3 & \beta_3 & 0 & 0 \end{pmatrix}, \quad \text{with} \quad \beta_1, \beta_2, \beta_3 \in \mathbb{R}, \qquad (4)$$

$$T := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad R := \begin{pmatrix} \sigma_{v,1} & 0 & 0 \\ 0 & \sigma_{v,2} & 0 \\ 0 & 0 & \sigma_{v,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (5)$$

respectively, with $\sigma_{v,1}, \sigma_{v,2}, \sigma_{v,3} \in \mathbb{R}^+$.

Nonlinear state-space model

Consider the nonlinear state-space model for the GST (LG, Ortega, Peresetsky [2015]):

$$\begin{cases} \mathbf{r}_t = \boldsymbol{\alpha} + B\mathbf{e}_t + \mathbf{u}_t, & \{\mathbf{u}_t\} \sim \mathrm{WN}\left(\mathbf{0}_3, \Sigma_u\right), \quad (6a) \\ \mathbf{e}_t = T\mathbf{e}_{t-1} + R_{t-1}(\mathbf{e}_{t-1})\mathbf{v}_{t-1}, & \{\mathbf{v}_t\} \sim \mathrm{WN}(\mathbf{0}_3, \mathbb{I}_3), \quad (6b) \end{cases}$$

where $\mathbf{e}_t \in \mathbb{R}^5$, $\boldsymbol{\alpha} \in \mathbb{R}^3$, $\Sigma_u \in \mathbb{D}_3$, $B \in \mathbb{M}_{5,3}$, $T \in \mathbb{M}_5$ are as in the linear state-space model (3a)-(3b), and $R \in \mathbb{M}_{3,5}$, is defined as

$$R_{t-1}(\mathbf{e}_{t-1}) := \begin{pmatrix} \sigma_{1,t}(\mathbf{e}_{t-1}) & 0 & 0 \\ 0 & \sigma_{2,t}(\mathbf{e}_{t-1}) & 0 \\ 0 & 0 & \sigma_{3,t}(\mathbf{e}_{t-1}) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (7)$$

where functional dependences for $\sigma_{i,t}(\mathbf{e}_{t-1})$, $i \in \{1, 2, 3\}$ are chosen to be consistent with the chronology of the intra-day quoting of the components of the GST ε_t.

Nonlinear state-space model

Additionally for the state equation

$$\mathbf{e}_t = T\mathbf{e}_{t-1} + R_{t-1}(\mathbf{e}_{t-1})\mathbf{v}_{t-1}, \quad \{\mathbf{v}_t\} \sim \mathrm{WN}(\mathbf{0}_3, \mathbb{I}_3)$$

we set

$$Q_t := R_{t-1}(\mathbf{e}_{t-1})R_{t-1}(\mathbf{e}_{t-1})^\top, \tag{8}$$

and hence

$$Q_t = \left(\begin{array}{c|ccc} \Sigma_{v,t} & & & \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \text{ with } \Sigma_{v,t} := \begin{pmatrix} \sigma^2_{1,t}(\mathbf{e}_{t-1}) & 0 & 0 \\ 0 & \sigma^2_{2,t}(\mathbf{e}_{t-1}) & 0 \\ 0 & 0 & \sigma^2_{3,t}(\mathbf{e}_{t-1}) \end{pmatrix}. \tag{9}$$

The choice of these specific matrix forms for $R_{t-1}(\mathbf{e}_{t-1})$ and Q_t allows to:

- Use the extended Kalman filter (EKF) to estimate the GST components ε_t.
- Use the dynamic model for the description and the forecasting of the conditional variances $\sigma_{i,t}(\mathbf{e}_{t-1})$, $i \in \{1, 2, 3\}$, while neglecting the correlation between the GST components.

Linear and extended Kalman estimation and filtering

The general recursive scheme for the linear (LKF) and extended Kalman filters (EKF) associated to the linear and nonlinear models, respectively, is:

$$\mathbf{u}_t = \mathbf{r}_t - (\boldsymbol{\alpha} + B\boldsymbol{\epsilon}_t), \tag{10}$$

$$H_t = BP_tB^\top + \Sigma_u, \tag{11}$$

$$\boldsymbol{\epsilon}_{t|t} = \boldsymbol{\epsilon}_t + K_t\mathbf{u}_t, \text{ with } K_t := P_tB^\top H_t^{-1}, \tag{12}$$

$$P_{t|t} = P_t - K_tBP_t, \tag{13}$$

$$\boldsymbol{\epsilon}_{t+1} = T\boldsymbol{\epsilon}_{t|t}, \tag{14}$$

$$P_{t+1} = TP_{t|t}T^\top + Q_t, \text{ with } Q_t := R_t(\boldsymbol{\epsilon}_{t|t})R_t(\boldsymbol{\epsilon}_{t|t})^\top, \tag{15}$$

where $\mathbf{r} := \{\mathbf{r}_1, \ldots, \mathbf{r}_T\}$ is the sample, $\boldsymbol{\epsilon}_{t|t} := \mathrm{E}[\mathbf{e}_t|\mathcal{F}_t]$ (respectively, $\boldsymbol{\epsilon}_{t+1} := \mathrm{E}[\mathbf{e}_{t+1}|\mathcal{F}_t]$) are the **updated or filtered** (respectively, **forecasted**) state vectors, $P_{t|t} := \mathrm{E}\left[\mathbf{e}_t\mathbf{e}_t^\top|\mathcal{F}_t\right]$ (respectively, $P_{t+1} := \mathrm{E}\left[\mathbf{e}_{t+1}\mathbf{e}_{t+1}^\top|\mathcal{F}_t\right]$) are the corresponding covariance matrices; $\mathcal{F}_t = \sigma(\mathbf{r}_1, \ldots, \mathbf{r}_t)$.

Specifications for the conditional variances in the nonlinear state-space model

Model 1 for $\sigma_{i,t}(\mathbf{e}_{t-1})$, $i \in \{1,2,3\}$:

$$\sigma_{1,t}^2 = a_1 + \delta_1 \sigma_{3,t-1}^2 + \gamma_1 \varepsilon_{3,t-1}^2, \tag{16}$$

$$\sigma_{2,t}^2 = a_2 + \delta_2 \sigma_{1,t}^2, \tag{17}$$

$$\sigma_{3,t}^2 = a_3 + \delta_3 \sigma_{2,t}^2, \tag{18}$$

with the parameter constraints: $\gamma_1 \geq 0$ and $a_i > 0$, $\delta_i \geq 0$, $i \in \{1,2,3\}$ and $\delta_2\delta_3(\delta_1 + \gamma_1) < 1$.

Model 2 for $\sigma_{i,t}(\mathbf{e}_{t-1})$, $i \in \{1,2,3\}$:

$$\sigma_{1,t}^2 = a_1 + \delta_1 \sigma_{3,t-1}^2 + \gamma_1 \varepsilon_{3,t-1}^2 + \rho_1 \sigma_{1,t-1}^2 + \tau_1 \varepsilon_{1,t-1}^2, \tag{19}$$

$$\sigma_{2,t}^2 = a_2 + \delta_2 \sigma_{1,t}^2 + \rho_2 \sigma_{2,t-1}^2 + \tau_2 \varepsilon_{2,t-1}^2, \tag{20}$$

$$\sigma_{3,t}^2 = a_3 + \delta_3 \sigma_{2,t}^2 + \rho_3 \sigma_{3,t-1}^2 + \tau_3 \varepsilon_{3,t-1}^2, \tag{21}$$

with the parameter constraints: $\gamma_1 \geq 0$ and $a_i > 0$, $\delta_i, \rho_i, \tau_i \geq 0$, $i \in \{1,2,3\}$, and

$$\begin{cases} (\rho_1 + \tau_1)(1 + \delta_2(1 + \delta_3)) < 1, & \text{(22a)} \\ (\rho_2 + \tau_2)(1 + \delta_2(1 + \delta_3)) < 1, & \text{(22b)} \\ (\delta_1 + \gamma_1)(1 + \delta_2(1 + \delta_3)) + \tau_3 + \rho_3 < 1. & \text{(22c)} \end{cases}$$

Identification of the models

Proposition

- *The linear state-space model* (3a)-(3b) *is well identified whenever one of the elements of* $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)^\top$ *is set equal to a constant or, alternatively, when one of the unconditional variances* $\sigma^2_{v,1}, \sigma^2_{v,2}, \sigma^2_{v,3}$ *is set equal to a positive constant.*
- *The nonlinear state-space model* (6a)-(6b) *is well identified whenever one of the elements of the vector* $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)^\top$ *is set equal to a constant or, alternatively, when one of the components of the vector* $\mathbf{a} := (a_1, a_2, a_3)^\top$ *that determine the models 1 or 2 is set equal to a positive constant.*

Results of the LKF and EKF estimation

In the case of the linear state-space model the LKF yields:

- the filtered $\widehat{\varepsilon_t} := \boldsymbol{\epsilon}_{t|t}$ of the GST up to time t
- the estimated $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$, $\sigma_{u,1}, \sigma_{u,2}, \sigma_{u,3}, \sigma_{v,1}, \sigma_{v,2}, \sigma_{v,3} \in \mathbb{R}^+$
- the model is intrinsically homoscedastic: $H_t := \mathrm{E}[(\mathbf{r}_t - \boldsymbol{\alpha})(\mathbf{r}_t - \boldsymbol{\alpha})^\top \mid \mathcal{F}_{t-1}]$ and $P_t := \mathrm{E}\left[\mathbf{e}_t \mathbf{e}_t^\top \mid \mathcal{F}_{t-1}\right]$ are asymptotically constant in time

In the case of the nonlinear state-space model the EKF yields:

- the filtered $\widehat{\varepsilon_t} := \boldsymbol{\epsilon}_{t|t}$ of the GST up to time t and the estimated $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$, $\sigma_{u,1}, \sigma_{u,2}, \sigma_{u,3} \in \mathbb{R}^+$
- for the conditional variances $\sigma_{i,t}(\mathbf{e}_{t-1})$, $i \in \{1,2,3\}$ we get the estimated γ_1, $a_i > 0$, δ_i, $i \in \{1,2,3\}$ in the case of Model 1, and, additionally ρ_i , τ_i, $i \in \{1,2,3\}$ for the Model 2
- the covariance matrices Q_t and P_t have temporal dynamics but imply the zero correlation between the GST components

Volatility forecasting: extended filtration

Observation: linear/nonlinear state-space models of the GST are not complete to be used for volatility forecasting.

Suggestion: use multivariate heteroscedastic models on the filtered $\widehat{\varepsilon}_t$ of the GST $\{\varepsilon_t\}$ and on the residuals $\{\mathbf{u}_t\}$.

Introduce the **extended filtration** with

$$\mathcal{F}_t^* := \sigma\left(\{\mathbf{r}_1, \ldots, \mathbf{r}_t\} \cup \{\widehat{\varepsilon}_1, \ldots, \widehat{\varepsilon}_t\}\right),$$

the pseudo-information sets generated by the observed returns $\{\mathbf{r}_1, \ldots, \mathbf{r}_t\}$ and the LKF/EKF filtered $\widehat{\varepsilon}_t := \boldsymbol{\epsilon}_{t|t}$ of the GST up to time $t \in \{1, \ldots, T\}$.

Use the **chronologically adapted models** for $\{\mathcal{F}_t^*\}$-predictable $\{P_t^*\}$ and $\{\Sigma_{u,t}^*\}$ s.t.:

$$\widehat{\mathbf{e}}_t|_{\mathcal{F}_{t-1}^*} \sim \mathrm{WN}(\mathbf{0}_5, P_t^*) \quad \text{and} \quad \mathbf{u}_t|_{\mathcal{F}_{t-1}^*} \sim \mathrm{WN}(\mathbf{0}_3, \Sigma_{u,t}^*).$$

Volatility forecast based on $\mathcal{F}_{t-1}^*$

$$\underline{H_t^* := \mathrm{E}[(\mathbf{r}_t - \boldsymbol{\alpha})(\mathbf{r}_t - \boldsymbol{\alpha})^\top | \mathcal{F}_{t-1}^*] = B P_t^* B^\top + \Sigma_{u,t}^*} \tag{23}$$

GST-based volatility forecasting using the non-/linear state-space model: DCC models for $\{\widehat{\boldsymbol{\varepsilon}_t}\}$ and for $\{\mathbf{u}_t\}$

Let $\zeta_{i,t} := \widehat{\varepsilon_{i,t}}/\sigma_{i,t}\left(\widehat{\mathbf{e}_{t-1}}\right)$ with $\sigma_{i,t}\left(\widehat{\mathbf{e}_{t-1}}\right)$, $i = \{1,2,3\}$ based on Model 1 (16)-(18) or on Model 2 (33)-(33). Let $D_t := \operatorname{diag}\left(\sigma_{1,t}\left(\widehat{\mathbf{e}_{t-1}}\right), \sigma_{2,t}\left(\widehat{\mathbf{e}_{t-1}}\right), \sigma_{3,t}\left(\widehat{\mathbf{e}_{t-1}}\right)\right)$.

The **Hadamard DCC model** for $\boldsymbol{\zeta}_t$ is:

$$\Sigma_t = D_t R_t D_t, \tag{24}$$

$$R_t = Q_t^{*-1/2} Q_t Q_t^{*-1/2}, \quad Q_t^* := \mathbb{I}_3 \odot Q_t, \tag{25}$$

$$Q_t = (\mathbf{i}_3\mathbf{i}_3^\top - A - B) \odot Q + A \odot (\boldsymbol{\zeta}_{t-1}\boldsymbol{\zeta}_{t-1}^\top) + B \odot Q_{t-1}. \tag{26}$$

The **scalar subfamily** is

$$Q_t = (1 - a - b)Q + a(\boldsymbol{\zeta}_{t-1}\boldsymbol{\zeta}_{t-1}^\top) + bQ_{t-1}. \tag{27}$$

In both cases the model parameters are subjected to the positivity and stationarity constraints, respectively.

Volatility forecast based on nonlinear state-space models combined with DCC models

The DCC model yields

$$\Sigma^*_{\widehat{\varepsilon},t} := \mathrm{E}\left[\widehat{\boldsymbol{\varepsilon}}_t \widehat{\boldsymbol{\varepsilon}}_t^{\top} \mid \mathcal{F}^*_{t-1}\right] = D_t R_t D_t, \tag{28}$$

$$P^*_t := \mathrm{E}\left[\widehat{\mathbf{e}}_t \widehat{\mathbf{e}}_t^{\top} \mid \mathcal{F}^*_{t-1}\right] = \left(\begin{array}{c|cc} \Sigma^*_{\widehat{\varepsilon},t} & 0 & \\ \hline 0 & \widehat{\varepsilon}^2_{2,t-1} & \widehat{\varepsilon}_{2,t-1}\widehat{\varepsilon}_{3,t-1} \\ & \widehat{\varepsilon}_{2,t-1}\widehat{\varepsilon}_{3,t-1} & \widehat{\varepsilon}^2_{3,t-1} \end{array}\right) \tag{29}$$

Analogously the conditional covariances $\{\Sigma^*_{u,t}\}$ of the residuals $\{\mathbf{u}_t\}$ are described by another DCC model.

Hence

$$\underline{H^*_{t|t-1} := \mathrm{E}[(\mathbf{r}_t - \boldsymbol{\alpha})(\mathbf{r}_t - \boldsymbol{\alpha})^{\top} \mid \mathcal{F}^*_{t-1}] = BP^*_t B^{\top} + \Sigma^*_{u,t}.}$$

▸ Compare with the linear case

GST-based volatility forecasting using the linear state-space model

We use the same approach but unlike in the nonlinear state-space model setup, the conditional variances of the GST and of the residuals for the linear state-space model are not determined. We need to model them separately.

Models for the conditional variances of the GST and of the residuals for the linear state-space model

The intraday dependence can be used unlike in the nonlinear state-space model

Model 1 for the conditional variances:

$$\sigma^2_{1,t} = a_1 + \delta_1 \sigma^2_{3,t-1} + \gamma_1 \widehat{\varepsilon}^2_{3,t-1}, \tag{30}$$

$$\sigma^2_{2,t} = a_2 + \delta_2 \sigma^2_{1,t} + \gamma_2 \widehat{\varepsilon}^2_{1,t}, \tag{31}$$

$$\sigma^2_{3,t} = a_3 + \delta_3 \sigma^2_{2,t} + \gamma_3 \widehat{\varepsilon}^2_{2,t}. \tag{32}$$

Model 2 for the conditional variances:

$$\sigma^2_{1,t} = a_1 + \delta_1 \sigma^2_{3,t-1} + \gamma_1 \widehat{\varepsilon}^2_{3,t-1} + \rho_1 \sigma^2_{1,t-1} + \tau_1 \widehat{\varepsilon}^2_{1,t-1}, \tag{33}$$

$$\sigma^2_{2,t} = a_2 + \delta_2 \sigma^2_{1,t} + \gamma_2 \widehat{\varepsilon}^2_{1,t} + \rho_2 \sigma^2_{2,t-1} + \tau_2 \widehat{\varepsilon}^2_{2,t-1}, \tag{34}$$

$$\sigma^2_{3,t} = a_3 + \delta_3 \sigma^2_{2,t} + \gamma_3 \widehat{\varepsilon}^2_{2,t} + \rho_3 \sigma^2_{3,t-1} + \tau_3 \widehat{\varepsilon}^2_{3,t-1}. \tag{35}$$

Intraday extended filtrations in the linear models

For each trading date t we use three different **intraday extended information sets**:

- $\mathcal{F}^*_{t_1} := \sigma\left(\{\mathbf{r}_1, \ldots, \mathbf{r}_t\} \cup \{\widehat{\boldsymbol{\varepsilon}}_1, \ldots, \widehat{\boldsymbol{\varepsilon}}_{t-1}, \widehat{\varepsilon}_{t_1}\}\right)$
- $\mathcal{F}^*_{t_2} := \sigma\left(\{\mathbf{r}_1, \ldots, \mathbf{r}_t\} \cup \{\widehat{\boldsymbol{\varepsilon}}_1, \ldots, \widehat{\boldsymbol{\varepsilon}}_{t-1}, \widehat{\varepsilon}_{t_1}, \widehat{\varepsilon}_{t_2}\}\right)$
- $\mathcal{F}^*_{t_3} := \mathcal{F}^*_t = \sigma\left(\{\mathbf{r}_1, \ldots, \mathbf{r}_t\} \cup \{\widehat{\boldsymbol{\varepsilon}}_1, \ldots, \widehat{\boldsymbol{\varepsilon}}_{t-1}\}\right)$

▸ Diagram

Notice that in the nonlinear setup only $\mathcal{F}^*_t$ is used due to the requirements of the extended Kalman filter framework.

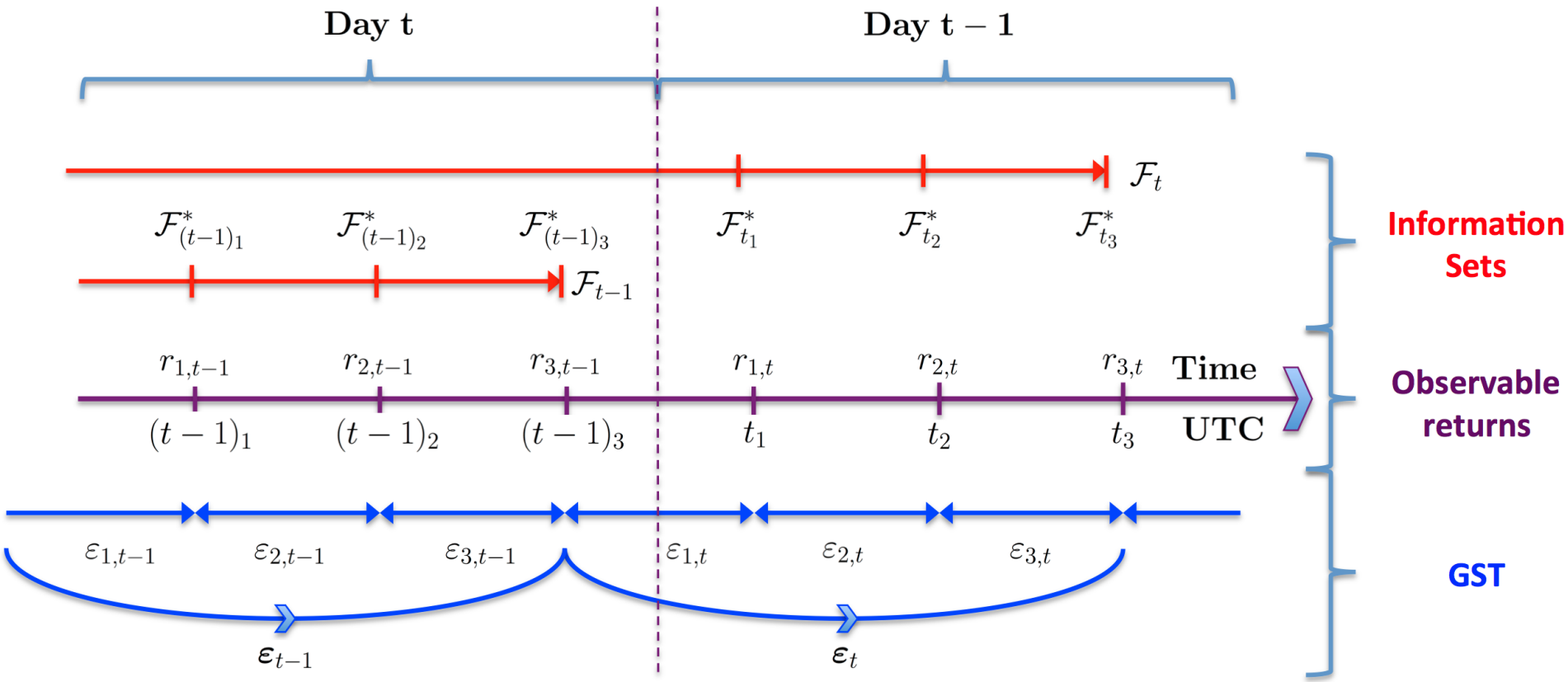


Figure 2 : The chronology of the variables, time labels, and information sets.

Back

Positivity and stationarity of the conditional variance models

We write Model 1 and Model 2 in the form of the one-dimensional **tvGARCH(1,1)** and **tvGARCH(3,3)** models, respectively [DR06, CS09, RR13]:

$$\sigma^2_{t_i} = a_{t_i} + \delta_{t_i}\sigma^2_{t_i-1} + \gamma_{t_i}\widehat{\varepsilon}^2_{t_i-1}, \tag{36}$$

$$\sigma^2_{t_i} = a_{t_i} + \delta_{t_i}\sigma^2_{t_i-1} + \gamma_{t_i}\widehat{\varepsilon}^2_{t_i-1} + \rho_{t_i}\sigma^2_{t_i-3} + \tau_{t_i}\widehat{\varepsilon}^2_{t_i-3}, \tag{37}$$

with $i \in \{1,2,3\}$, $a_{t_i} := a_i$, $\delta_{t_i} := \delta_i$, $\gamma_{t_i} := \gamma_i$, $\rho_{t_i} := \rho_i$, $\tau_{t_i} := \tau_i$, and

$$t_i - 1 := \begin{cases} (t-1)_3 & \text{if} \quad i = 1, \\ t_{i-1} & \text{if} \quad i \in \{2,3\}. \end{cases}$$

Positivity constraints: all the coefficients > 0.
Standard sufficient stationarity constraints [RR13]: $\delta_{t_i} + \gamma_{t_i} < 1$ for the Model 1, and $\delta_{t_i} + \gamma_{t_i} + \rho_{t_i} + \tau_{t_i} < 1$ for the Model 2, with $i \in \{1,2,3\}$, LACK SHARPNESS.

Improved stationarity conditions

Introduce the matrices A_{t_i} associated to the Markov representations of (36) and $A_i := \mathrm{E}[A_{t_i}]$ (see [FZ10] for the details). Let $\{\mathbf{v}_t\} \sim \mathrm{WN}(\mathbf{0}_3, \mathbb{I}_3)$ be the innovations introduced in the in the linear state-space model (3b). Then:

$$A_{t_i} := \begin{pmatrix} \gamma_{t_i} v_{t_i}^2 & 0 & \tau_{t_i} v_{t_i}^2 & \delta_{t_i} v_{t_i}^2 & 0 & \rho_{t_i} v_{t_i}^2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \gamma_{t_i} & 0 & \tau_{t_i} & \delta_{t_i} & 0 & \rho_{t_i} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad A_i := \begin{pmatrix} \gamma_i & 0 & \tau_i & \delta_i & 0 & \rho_i \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \gamma_i & 0 & \tau_i & \delta_i & 0 & \rho_i \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Proposition

Consider the GARCH models with time-varying coefficients defined by the tvGARCH recursions (36)-(37). If the innovations $\{\mathbf{v}_t\}$ that drive them are independent, then the following conditions imply the existence of a unique stationary solution:

(i) *For Model 1:* $(\delta_1 + \gamma_1)(\delta_2 + \gamma_2)(\delta_3 + \gamma_3) < 1$.

(ii) *For Model 2:* $\rho(A_3 A_2 A_1) < 1$.

Forecasting using the intraday extended information sets

The DCC models yield:

$$\widehat{\boldsymbol{\varepsilon}}_t = L_t^{\widehat{\boldsymbol{\varepsilon}}} \boldsymbol{\xi}_t^{\widehat{\boldsymbol{\varepsilon}}} \quad \text{with} \quad \{\boldsymbol{\xi}_t^{\widehat{\boldsymbol{\varepsilon}}}\}, \sim \mathrm{IN}\left(\mathbf{0}, \mathbb{I}_3\right),$$

where $L_t^{\widehat{\boldsymbol{\varepsilon}}}$ is the lower triangular Cholesky factor of $H_t^{\widehat{\boldsymbol{\varepsilon}}}$.

(i) **Forecasting with respect to** $\mathcal{F}_{t_2}^*$: $\mathrm{var}\left(\widehat{\boldsymbol{\varepsilon}}_t \mid \mathcal{F}_{t_2}^*\right) = \mathrm{diag}\left(\widehat{\varepsilon}_{1,t}^2, \widehat{\varepsilon}_{2,t}^2, \sigma_{3,t}^{\widehat{\boldsymbol{\varepsilon}}\,2}\right)$,

$$\mathrm{E}\left[\widehat{\boldsymbol{\varepsilon}}_t \widehat{\boldsymbol{\varepsilon}}_t^\top \mid \mathcal{F}_{t_2}^*\right] = L_t^{\widehat{\boldsymbol{\varepsilon}}} \begin{pmatrix} \xi_{1,t}^{\widehat{\boldsymbol{\varepsilon}}\,2} & \xi_{1,t}^{\widehat{\boldsymbol{\varepsilon}}} \xi_{2,t}^{\widehat{\boldsymbol{\varepsilon}}} & 0 \\ \xi_{1,t}^{\widehat{\boldsymbol{\varepsilon}}} \xi_{2,t}^{\widehat{\boldsymbol{\varepsilon}}} & \xi_{2,t}^{\widehat{\boldsymbol{\varepsilon}}\,2} & 0 \\ 0 & 0 & 1 \end{pmatrix} L_t^{\widehat{\boldsymbol{\varepsilon}}\,\top} =: \boldsymbol{\Sigma}_{\widehat{\boldsymbol{\varepsilon}},t|t_2}^*$$

(ii) **Forecasting with respect to** $\mathcal{F}_{t_1}^*$ $(\gamma_3 = 0)$: $\mathrm{var}\left(\widehat{\boldsymbol{\varepsilon}}_t \mid \mathcal{F}_{t_1}^*\right) = \mathrm{diag}\left(\widehat{\varepsilon}_{1,t}^2, \sigma_{2,t}^{\widehat{\boldsymbol{\varepsilon}}\,2}, \sigma_{3,t}^{\widehat{\boldsymbol{\varepsilon}}\,2}\right)$,

$$\mathrm{E}\left[\widehat{\boldsymbol{\varepsilon}}_t \widehat{\boldsymbol{\varepsilon}}_t^\top \mid \mathcal{F}_{t_1}^*\right] = L_t^{\widehat{\boldsymbol{\varepsilon}}} \begin{pmatrix} \xi_{1,t}^{\widehat{\boldsymbol{\varepsilon}}\,2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} L_t^{\widehat{\boldsymbol{\varepsilon}}\,\top} =: \boldsymbol{\Sigma}_{\widehat{\boldsymbol{\varepsilon}},t|t_1}^*.$$

(iii) **Forecasting with respect to** $\mathcal{F}_{(t-1)_3}^* = \mathcal{F}_{t-1}^*$ $(\gamma_2 = \gamma_3 = 0)$:

$$\mathrm{var}\left(\widehat{\boldsymbol{\varepsilon}}_t \mid \mathcal{F}_{(t-1)_3}^*\right) = \mathrm{diag}\left(\sigma_{1,t}^{\widehat{\boldsymbol{\varepsilon}}\,2}, \sigma_{2,t}^{\widehat{\boldsymbol{\varepsilon}}\,2}, \sigma_{3,t}^{\widehat{\boldsymbol{\varepsilon}}\,2}\right),$$

$$\mathrm{E}\left[\widehat{\boldsymbol{\varepsilon}}_t \widehat{\boldsymbol{\varepsilon}}_t^\top \mid \mathcal{F}_{(t-1)_3}^*\right] = L_t^{\widehat{\boldsymbol{\varepsilon}}} L_t^{\widehat{\boldsymbol{\varepsilon}}\,\top} = H_t^{\widehat{\boldsymbol{\varepsilon}}} =: \boldsymbol{\Sigma}_{\widehat{\boldsymbol{\varepsilon}},t|(t-1)_3}^*.$$

Volatility forecast based on linear state-space models combined with DCC models

The DCC models yield $\Sigma^*_{\widehat{\varepsilon},t|t_i}$.

$$P^*_{t|t_i} := \mathrm{E}\left[\widehat{\mathbf{e}}_t\widehat{\mathbf{e}}_t^{\top} \mid \mathcal{F}^*_{t_i}\right] = \left(\begin{array}{c|cc} \Sigma^*_{\widehat{\varepsilon},t|t_i} & \multicolumn{2}{c}{0} \\ \hline 0 & \widehat{\varepsilon}^2_{2,t-1} & \widehat{\varepsilon}_{2,t-1}\widehat{\varepsilon}_{3,t-1} \\ & \widehat{\varepsilon}_{2,t-1}\widehat{\varepsilon}_{3,t-1} & \widehat{\varepsilon}^2_{3,t-1} \end{array}\right). \tag{38}$$

Analogously the conditional covariances $\{\Sigma^*_{u,t}\}$ of the residuals $\{\mathbf{u}_t\}$ are described by another DCC model.

Hence

$$\underline{H^*_{t|t_i} := \mathrm{E}[(\mathbf{r}_t - \boldsymbol{\alpha})(\mathbf{r}_t - \boldsymbol{\alpha})^{\top} \mid \mathcal{F}^*_{t_i}] = BP^*_{t|t_i}B^{\top} + \Sigma^*_{u,t|t_i}}.$$

▸ Compare with the nonlinear case

Empirical study: the dataset

Dataset[a]: The daily closing values of the 3 stock market indices with asynchronous closing times:

- NIKKEI 225(6:00 UTC)
- FTSE(16:30 UTC)
- S&P500(21:00 UTC)

Total period: January 5, 1996 – April 1, 2015; $T := 4581$ observations.

Estimation period: January 5, 1996 – December 4, 2006; $T_{\text{est}} := 2600$.

Out-of-sample period: December 5, 2006 – April 1, 2015; $T_{\text{out}} := 1981$.

[a] Data were downloaded from the Yahoo Finance database

MCS: Covariance loss functions

Depending on the specific intraday extended information set used for forecasting:

$$d^{\text{COV}}_{\mathcal{F}^*_{t_2}} := \left(r^2_{3,t} - h_{33,t}\right)^2, \tag{39}$$

$$d^{\text{COV}}_{\mathcal{F}^*_{t_1}} := \frac{1}{3} \sum_{i \leq j = 2,3} \left(r_{i,t} r_{j,t} - h_{ij,t}\right)^2, \tag{40}$$

$$d^{\text{COV}}_{\mathcal{F}^*_{(t-1)_3}} := \frac{1}{6} \sum_{i \leq j = 1,2,3} \left(r_{i,t} r_{j,t} - h_{ij,t}\right)^2, \tag{41}$$

where $t \in \{T_{\text{est}} + 1, \ldots, T_{\text{est}} + T_{\text{out}}\}$ and $h_{ij,t}$ are the (i,j)-entries of the model dependent forecasts for the conditional covariance matrices at t, namely:

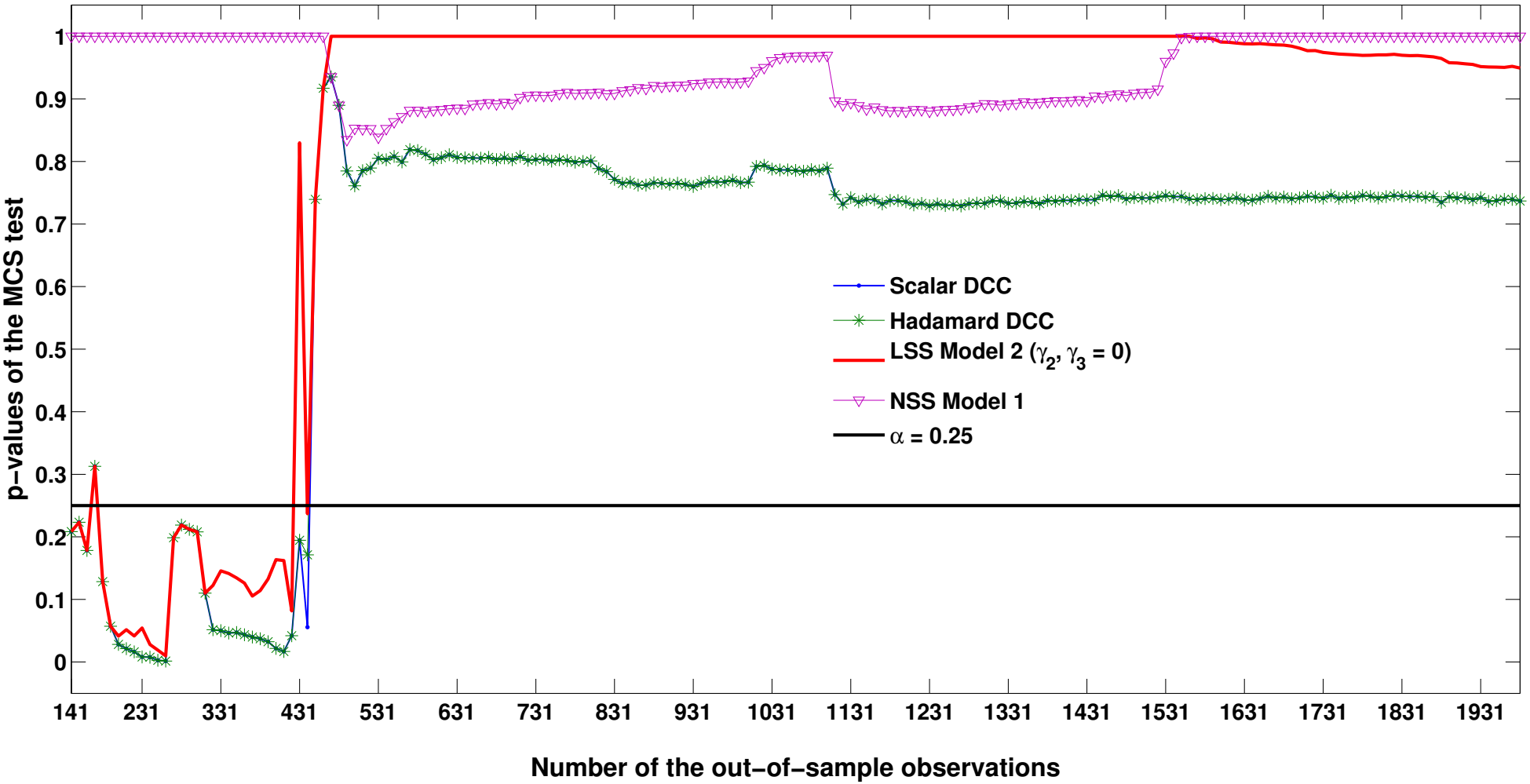
- Benchmark scalar/Hadamard DCC models: H_t w.r.t. the information set $\mathcal{F}_{t-1}$
- Nonlinear state-space models: $H^*_{t|t-1}$ w.r.t. $\mathcal{F}^*_{t-1}$
- Linear state-space models: $H^*_{t|t_i}$ w.r.t. different intraday extended information sets

MCS: COV results

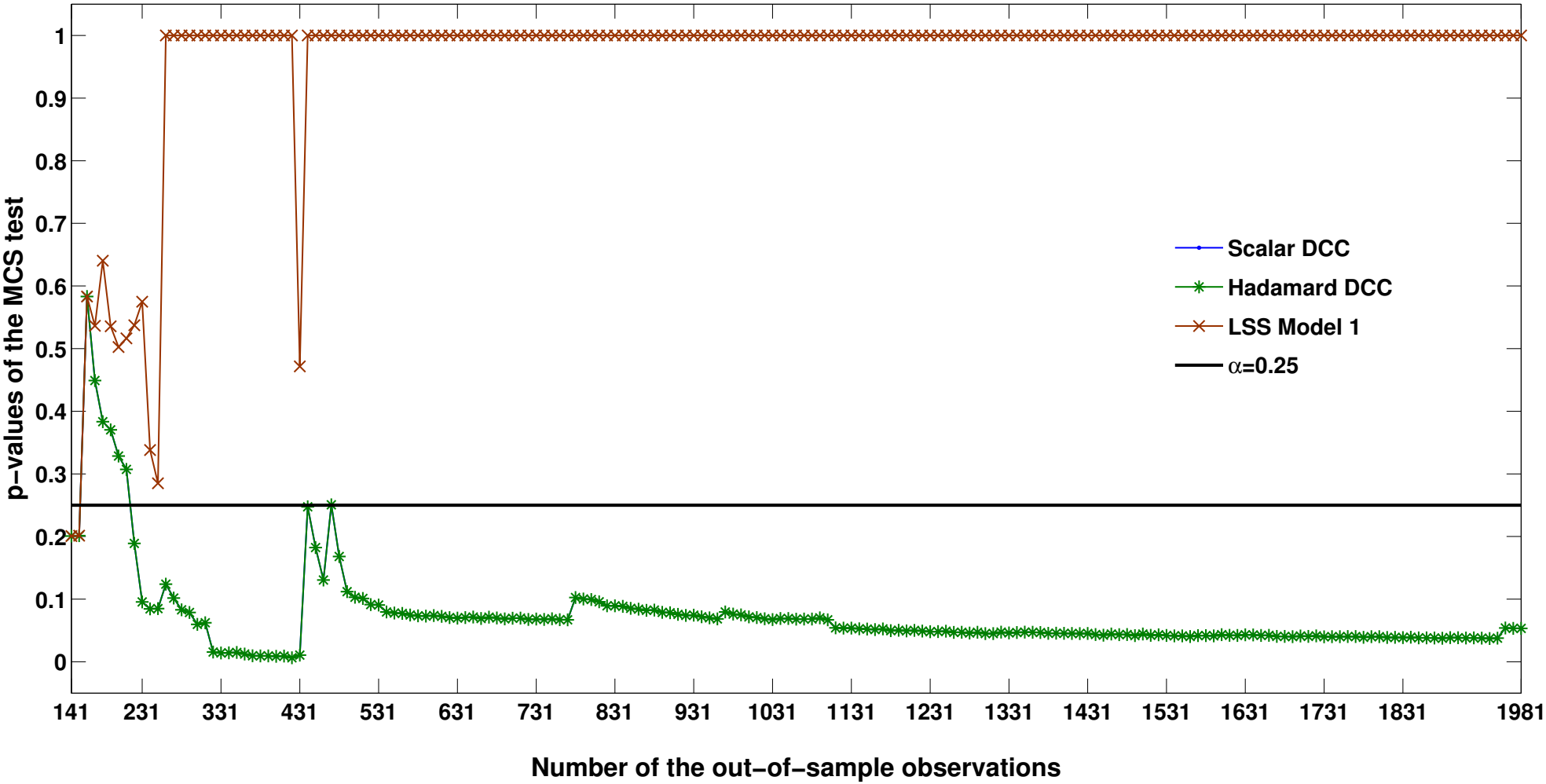
	DCC models		Linear state-space models						Nonlinear state-space model
	Scalar	Hadamard	Model 1	Model 1 ($\gamma_3 = 0$)	Model 1 ($\gamma_3, \gamma_2 = 0$)	Model 2	Model 2 ($\gamma_3 = 0$)	Model 2 ($\gamma_3, \gamma_2 = 0$)	Model 1
MCS $d^{\text{COV}}_{\mathcal{F}^*_{t_2}}$	7	7	**183**	29	1	33	28	2	6
Sum p-values	13.5594	13.5594	**177.9236**	22.0953	11.4554	28.9149	20.3553	11.1596	12.7184
MCS $d^{\text{COV}}_{\mathcal{F}^*_{t_1}}$	5	5	—	**185**	4	—	62	10	11
Sum p-values	14.3411	14.4063	—	**184.2549**	14.0456	—	45.4273	15.9334	16.0872
MCS $d^{\text{COV}}_{\mathcal{F}^*_{(t-1)_3}}$	155	155	—	—	1	—	—	156	**185**
Sum p-values	120.2530	120.3687	—	—	12.5903	—	—	157.3078	**174.6051**

Table 1 : MCS COV for $l(k) = T_{\text{est}} + 141 + 10k$, $k \in \{0, 1, \ldots, 184\}$. The MCS confidence level is set to 75%.

MCS COV for $d^{\text{COV}}_{\mathcal{F}^*_{(t-1)_3}}$: Evolution of the p-values



MCS COV for $d^{\mathrm{COV}}_{\mathcal{F}^*_{t_2}}$: Evolution of the p-values



MCS: Kullback-Leibler Information Criterion (KLIC) loss functions

$$d^{\mathrm{KLIC}}_{\mathcal{F}^*_{t_2}} := -\frac{1}{t-T_{\mathrm{est}}}\sum_{i=T_{\mathrm{est}}+1}^{t}\ln\left[\frac{1}{\sqrt{2\pi h_{33,i}}}\exp\left(-\frac{r^2_{3,i}}{2h_{33,i}}\right)\right], \tag{42}$$

$$d^{\mathrm{KLIC}}_{\mathcal{F}^*_{t_1}} := -\frac{1}{t-T_{\mathrm{est}}}\sum_{i=T_{\mathrm{est}}+1}^{t}\ln\left[\frac{1}{2\pi}\left[\det\begin{pmatrix} h_{22,i} & h_{23,i}\\ h_{32,i} & h_{33,i}\end{pmatrix}\right]^{-\frac{1}{2}}\times \right.$$
$$\left.\times\exp\left(-\frac{1}{2}\begin{pmatrix} r_{2,i}\\ r_{3,i}\end{pmatrix}^{\top}\begin{pmatrix} h_{22,i} & h_{23,i}\\ h_{32,i} & h_{33,i}\end{pmatrix}^{-1}\begin{pmatrix} r_{2,i}\\ r_{3,i}\end{pmatrix}\right)\right], \tag{43}$$

$$d^{\mathrm{KLIC}}_{\mathcal{F}^*_{(t-1)_3}} := -\frac{1}{t-T_{\mathrm{est}}}\sum_{i=T_{\mathrm{est}}+1}^{t}\ln\left[\frac{1}{(2\pi)^{3/2}\det(H_i)^{1/2}}\exp\left(-\frac{1}{2}\mathbf{r}_i^{\top}H_i^{-1}\mathbf{r}_i\right)\right], \tag{44}$$

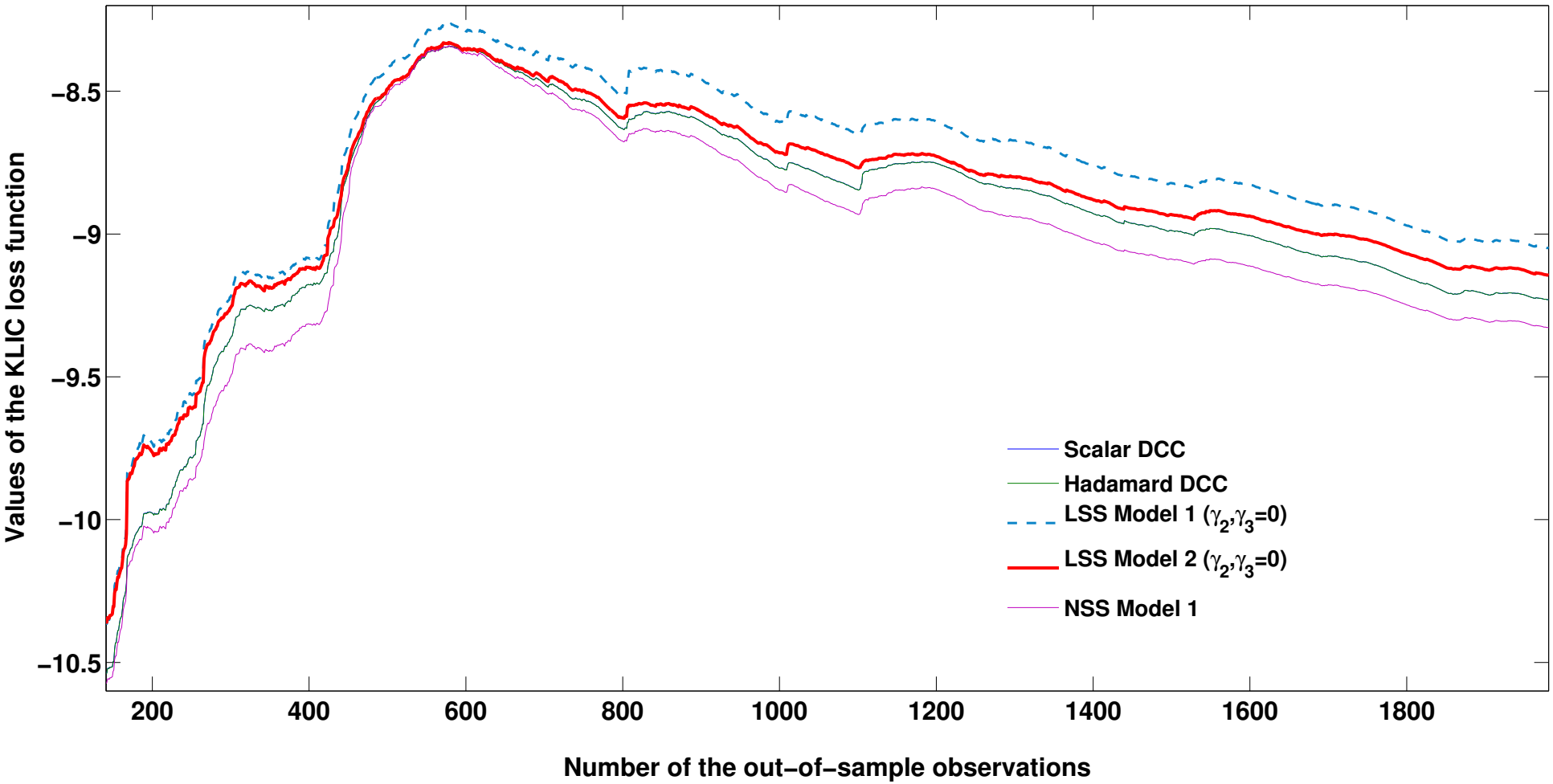
where $t \in \{T_{\mathrm{est}}+1,\ldots,T_{\mathrm{est}}+T_{\mathrm{out}}\}$ and $h_{ij,t}$ are the (i,j)-entries of the model dependent forecasts for the conditional covariance matrices at t computed as for the covariance loss functions.

MCS: KLIC results

	DCC models		Linear state-space models						Nonlinear state-space model
	Scalar	Hadamard	Model 1	Model 1 ($\gamma_3 = 0$)	Model 1 ($\gamma_3, \gamma_2 = 0$)	Model 2	Model 2 ($\gamma_3 = 0$)	Model 2 ($\gamma_3, \gamma_2 = 0$)	Model 1
MCS $d^{\mathrm{KLIC}}_{\mathcal{F}^*_{t_2}}$	18	23	0	0	0	**167**	0	0	0
Sum p-values	9.8125	22.7399	0	0	0	**165.9375**	0	0	0
MCS $d^{\mathrm{KLIC}}_{\mathcal{F}^*_{t_1}}$	0	3	—	0	0	—	**185**	0	0
Sum p-values	0	1.6073	—	0	0	—	**185.0000**	0	0
MCS $d^{\mathrm{KLIC}}_{\mathcal{F}^*_{(t-1)_3}}$	0	15	—	—	0	—	—	0	**175**
Sum p-values	0	13.3896	—	—	0	—	—	0	**175.0000**

Table 2 : MCS KLIC for $l(k) = T_{\mathrm{est}} + 141 + 10k$, $k \in \{0, 1, \ldots, 184\}$. MCS confidence level is et to75%.

MCS: values of the KLIC $d^{\mathrm{KLIC}}_{\mathcal{F}^*_{(t-1)_3}}$



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