Generalization of binomial coefficients to numbers on the nodes of graphs

Anna Khmelnitskaya[♯], Gerard van der Laan[†] Dolf Talman[‡]

[♯] Saint-Petersburg State University

[†] VU University, Amsterdam

[‡] Tilburg University

Higher School of Economics

Moscow

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Binomial coefficients

- For any two integers $n \geq 0$ and $0 \leq k \leq n$, the number of combinations of k elements from a given set of n objects is conventionally denoted by C_n^k or $\binom{n}{k}$ and

$$C_n^k = \frac{n!}{(n-k)!k!}.$$

  This number appears, in particular, as a coefficient in binomial expansions, from where it gets the name of a *binomial coefficient*.

- Arranging $C_n^0, \ldots, C_n^n$ from left to right in a row for successive values of n, we obtain a triangular array called *Pascal's triangle*.

n															
0								1							
1							1		1						
2						1		2		1					
3					1		3		3		1				
4				1		4		6		4		1			
5			1		5		10		10		5		1		
6		1		6		15		20		15		6		1	
7	1		7		21		35		35		21		7		1

Figure: The first eight rows ($n+0, \ldots, 7$) of Pascal's triangle.

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6		1		6		15		20		15		6		1	
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Binomial coefficients

The history of the binomial coefficients dates back to 200 BC, they have many nice properties, see for instance the websites:

- Wikipedia (13 pages): https://en.wikipedia.org/wiki/Pascals triangle

- Math is Fun: http://www.mathsisfun.com/pascals-triangle.html

- Wolfram MathWorld: http://mathworld.wolfram.com/PascalsTriangle.html

Properties

- If n is prime, then for any $k = 1, \ldots, n-1$, C_n^k is divisible by this prime.

  Moreover, for each $n > 0$ and $k = 0, \ldots, n-1$,

  $$\frac{C_n^k}{C_n^{k+1}} = \frac{k+1}{n-k},$$

  i.e., for any two consecutive binomial coefficients C_n^k and C_n^{k+1} in row n of Pascal's triangle their ratio is equal to the ratio of the number $k+1$ of the positions $0, \ldots, k$ in that row from the position k to the left and the number $n-k$ of the positions $k+1, \ldots, n$ in that row from the position $k+1$ to the right.

  For example, for $n = 6$ and $k = 1$, we have $\frac{C_6^1}{C_6^2} = \frac{6}{15} = \frac{2}{5} = \frac{1+1}{6-1}$.

0								1								
1							1		1							
2						1		2		1						
3					1		3		3		1					
4				1		4		6		4		1				
5			1		5		10		10		5		1			
6		1		6		15		20		15		6		1		
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Properties

- For any $n \geq 1$ and $0 \leq k \leq n$,

$$C_n^k = C_{n-1}^{k-1} + C_{n-1}^k,$$

with the convention that $C_{n-1}^{k-1} = 0$ if $k = 0$ and $C_{n-1}^k = 0$ if $k = n$.

n								
0	1							
1	1	1						
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Properties

- Let (n, k) denote position k on row $n \implies C_n^k$ is the number of paths in Pascal's triangle that start at $(0, 0)$ and terminate at (n, k), moving at every step downwards either to the left or to the right.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1^{*+} & & & & & & & \\
 & & & & & & 1^{*} & & 1^{+} & & & & & & \\
 & & & & & 1 & & 2^{*} & & 1^{+} & & & & & \\
 & & & & 1 & & 3^{*} & & 3 & & 1^{+} & & & & \\
 & & & 1 & & 4^{*} & & 6 & & 4^{+} & & 1 & & & \\
 & & 1 & & 5^{*} & & 10 & & 10^{+} & & 5 & & 1 & & \\
 & 1 & & 6 & & 15^{*} & & 20^{+} & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35^{*+} & & 35 & & 21 & & 7 & & 1
\end{array}
$$

Figure: Two of the paths from the apex $(0, 0)$ to position $(7, 3)$.

Obviously, C_n^k is also the number of paths from (n, k) to $(0, 0)$, moving upwards either to the left or to the right.

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1 \quad 3^{*} \quad 3 \quad 1^{+} \\
1 \quad 4^{*} \quad 6 \quad 4^{+} \quad 1 \\
1 \quad 5^{*} \quad 10 \quad 10^{+} \quad 5 \quad 1 \\
1 \quad 6 \quad 15^{*} \quad 20^{+} \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35^{*+} \quad 35 \quad 21 \quad 7 \quad 1
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For finite set N, a *graph* is a pair (N, E) with N the set of *nodes* and $E \subseteq \{\{i, j\} \mid i, j \in N, j \neq i\}$ a set of *edges* between nodes.

A graph (N, E) is connected if for any $i, j \in N$, $i \neq j$, there is a path from i to j in (N, E)

A node $k \in N$ is an *extreme node* of connected graph (N, E) if either $|N| = 1$, or $N \backslash \{k\}$ is connected in (N, E).

The set of extreme nodes of a connected graph (N, E) we denote by $S(N, E)$.

If $\{i, j\} \in E$, then node j is a *neighbor* of node i in (N, E).

Let $B_k^E = \{i \in N \mid \{i, k\} \in E\}$ denotee the set of neighbors of k in (N, E).

The number of neighbors of k in (N, E), denoted by $d_k(N, E)$, is *degree* of node k in (N, E), i.e., $d_k(N, E) = |B_k^E|$.

A connected graph (N, E) is a *line-graph*, or *chain*, if every node has at most two neighbors and $|E| = |N| - 1$.

For a graph (N, E) and node $i \in N$, we denote $N \backslash \{i\}$ by N_{-i} and $E|_{N_{-i}}$ by E_{-i}.

Feasible orderings

Given a finite set N, $\Pi(N)$ denotes the set of linear orderings on N.

For a connected graph (N, E) and node $k \in N$, a linear ordering $\pi \in \Pi(N)$, $\pi = (\pi_1, \ldots, \pi_{|N|})$, is *feasible* with respect to k in (N, E) if

(i) $\pi_1 = k$,

(ii) for $j = 2, \ldots, |N|$ the set of nodes $\{\pi_1, \ldots, \pi_j\}$ is connected in (N, E).

By $\Pi_k^E(N)$ we denote the subset of all feasible with respect to k in (N, E) linear orderings and its cardinality we denote by $c_k(N, E)$, i.e.,

$$c_k(N, E) = |\Pi_k^E(N)|.$$

Binomial coefficients revised on line-graphs

For integer $n \geq 0$, consider the $n+1$ positions on row n of Pascal's triangle as nodes on the line-graph (N, E) with $N = \{0, \ldots, n\}$ and $E = \{\{k, k+1\} \mid k = 0, \ldots, n-1\}$, where to every node k, $k = 0, \ldots, n$, the binomial coefficient C_n^k is assigned.

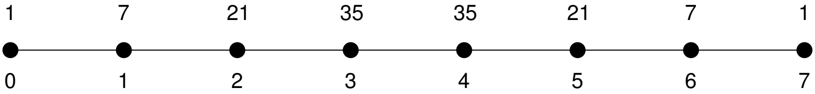

Figure: The binomial coefficients on the line-graph induced by row 7 of Pascal's triangle.

What are the numbers $c_k(N, E)$ for the nodes of this graph?

First, we give a new interpretation of C_n^k as the number of paths in Pascal's triangle from position (n, k) to apex $(0, 0)$.

For each linear ordering $\pi \in \Pi_k^E(N)$, for every $j = 2, \ldots, n+1$ node π_j is the neighbor of the node either on the left end or on the right end of the connected set $\{\pi_1, \ldots, \pi_{j-1}\}$.

$\Longrightarrow$ on the line-graph the number $c_k(N, E)$ of feasible orderings with respect to node k is equal to the number of paths from position (n, k) to $(0, 0)$ in Pascal's triangle.

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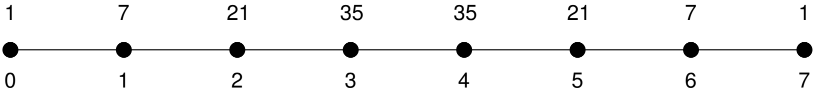

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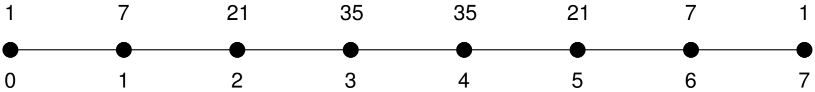

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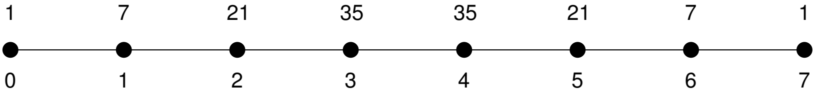

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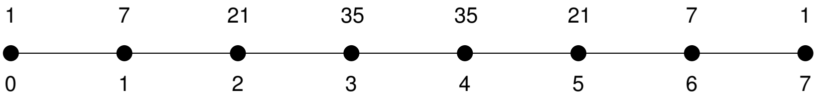

1^{*+}

1^{*} 1^{+}

1 2^{*} 1^{+}

1 3^{*} 3 1^{+}

1 4^{*} 6 4^{+} 1

1 5^{*} 10 10^{+} 5 1

1 6 15^{*} 20^{+} 15 6 1

1 7 21 35^{*+} 35 21 7 1

Two of the paths from position $(7, 3)$ to the apex $(0, 0)$:

path $^{+}$ corresponds to ordering $(3, 4, 5, 6, 7, 2, 1, 0)$ and path * to $(3, 2, 1, 4, 5, 6, 0, 7)$.

Theorem

For any two integers $n \geq 0$ and $0 \leq k \leq n$ it holds that for the line-graph (N, E) on $N = \{0, \ldots, n\}$ with $E = \{\{k, k+1\} \mid k = 0, \ldots, n-1\}$

$$C_n^k = |\{\pi \in \Pi(N) \mid \pi_1 = k, \{\pi_1, \ldots, \pi_j\} \text{ is connected in } (N, E),\ j = 2, \ldots, n+1\}|,$$

i.e., $C_n^k = c_k(N, E)$.

The theorem implies that C_n^k is equal to the number of ways the line-graph (N, E) can be constructed by starting with node k and adding at each step a node that is connected to one of the nodes that already have been added.

Equivalently, C_n^k is the total number of ways that extreme nodes can be removed one by one from the graph until only the node k remains.

Binomial coefficients revised on line-graphs

Now we reconsider formula $C_n^k = C_{n-1}^{k-1} + C_{n-1}^k$ within the framework of line-graphs.

For the line-graph (N, E) on $N = \{0, \ldots, n\}$ with $E = \{\{k, k+1\} \mid k = 0, \ldots, n-1\}$ consider the two line-subgraphs (N_{-0}, E_{-0}) and (N_{-n}, E_{-n}), both having n nodes and therefore corresponding to row $n-1$ of Pascal's triangle.

For every $k \in N_{-n} = \{0, \ldots, n-1\}$, $c_k(N_{-n}, E_{-n}) = |\Pi_k^{E_{-n}}(N_{-n})| = C_{n-1}^k$,
while for every $k \in N_{-0} = \{1, \ldots, n\}$, $c_k(N_{-0}, E_{-0}) = |\Pi_{k-1}^{E_{-0}}(N_{-0})| = C_{n-1}^{k-1}$.

Furthermore, we define $c_n(N_{-n}, E_{-n}) = 0$ and $c_0(N_{-0}, E_{-0}) = 0$.

$\Longrightarrow$ since $c_k(N, E) = |\Pi_k^E(N)| = C_n^k$, we obtain straightforwardly

Theorem

Let (N, E) be the line-graph with $N = \{0, \ldots, n\}$ and $E = \{\{k, k+1\} | k = 0, \ldots, n-1\}$ for some integer $n \geq 1$. Then for any integer $0 \leq k \leq n$ it holds that

$$c_k(N, E) = c_k(N_{-0}, E_{-0}) + c_k(N_{-n}, E_{-n}).$$

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$$c_k(N, E) = c_k(N_{-0}, E_{-0}) + c_k(N_{-n}, E_{-n}).$$

The theorem implies that the number of linear orderings feasible with respect to a node in the line-graph (N, E) is equal to the number of linear orderings feasible with respect to this node in the subgraph (N_{-0}, E_{-0}) without extreme node 0, plus the number of linear orderings feasible with respect to this node in the subgraph (N_{-n}, E_{-n}) without another extreme node n.

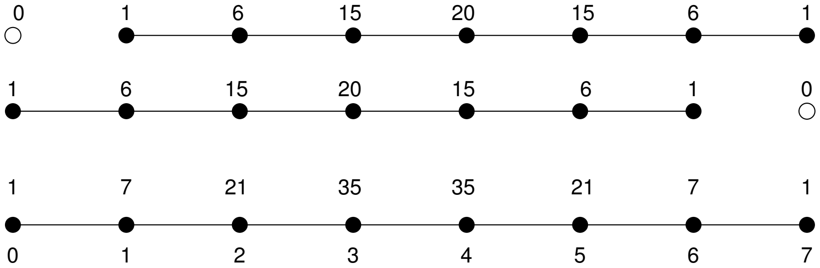

Figure: Illustration for $n = 7$ that the binomial coefficient of a node of a line-graph is equal to the sum of binomial coefficients of this node in two line-subgraphs without one of the extreme nodes.

For connected graph (N, E) and node $k \in N$, $c_k(N, E) = |\Pi_k^E(N)|$ is defined as the number of linear orderings π on N such that $\pi_1 = k$ and for $j = 2, \ldots, |N|$ the set $\{\pi_1, \ldots, \pi_j\}$ is connected in (N, E).

For a line-graph (N, E) these numbers are the binomial coefficients on row $|N| - 1$ in Pascal's triangle.

$\Longrightarrow$ we call these numbers Pascal graph numbers.

Definition

For a connected graph (N, E), the *Pascal graph number* of node $k \in N$ is the number $c_k(N, E)$.

For arbitrary connected graph (N, E), $c_k(N, E)$ is equal to the number of ways the graph can be constructed by starting with this node and adding at each step a node that is connected to one of the nodes that already have been added,

or equivalently, it is the number of ways extreme nodes can be removed from the graph one by one until only the node k remains.

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For a connected graph (N, E), the *Pascal graph number* of node $k \in N$ is the number $c_k(N, E)$.

For arbitrary connected graph (N, E), $c_k(N, E)$ is equal to the number of ways the graph can be constructed by starting with this node and adding at each step a node that is connected to one of the nodes that already have been added,

or equivalently, it is the number of ways extreme nodes can be removed from the graph one by one until only the node k remains.

A connected graph is *cycle-free* if for any $i, j \in N$, there is a unique path from i to j.

In a connected cycle-free graph (N, E) there are precisely $|N| - 1$ edges.

For connected cycle-free graph $(N.E)$ and node $k \in N$, a *satellite* N^E_{kh} of node k in (N, E) determined by neighbor $h \in B^E_k$ is the set of nodes $i \in N$ for which the unique path from node k to node i in (N, E) contains node h.

If (N, E) is cycle-free, each neighbor of k in (N, E) induces exactly one satellite of k,

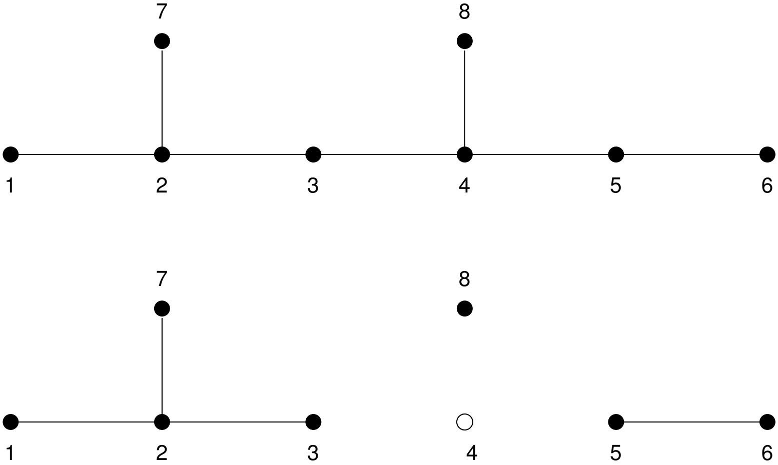
$\Longrightarrow$ the number of satellites of k in (N, E) is equal to $|B^E_k|$.

For every $k \in N$ the satellites of node k in (N, E) form a partition of N_{-k} and, therefore, $\sum_{h \in B^E_k} |N^E_{kh}| = |N| - 1$.

For any $k \in N$ and $h \in B^E_k$, we denote by (N^E_{kh}, E_{kh}) the subgraph of (N, E) on N^E_{kh}, where $E_{kh} = E|_{N^E_{kh}}$.

Each of these subgraphs is connected and cycle-free.

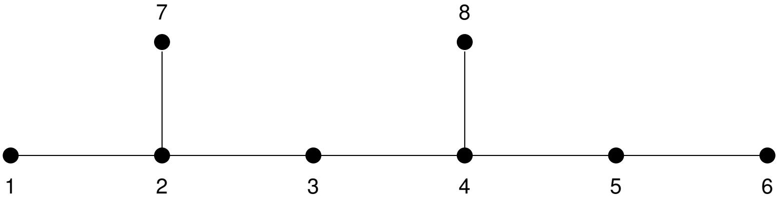
Pascal graph numbers on connected cycle-free graphs



For node 4, $B_4^E = \{3, 5, 8\}$ is the set of neighbors and $N_{43}^E = \{1, 2, 3, 7\}$, $N_{45}^E = \{5, 6\}$, and $N_{48}^E = \{8\}$ are the (three) satellites.

Pascal graph numbers on connected cycle-free graphs

A connected cycle-free graph (N, E) with $|N| \geq 2$ has at least two extreme nodes and moreover a node is an extreme node of (N, E) if and only if it has precisely one neighbor in (N, E).



$S(N, E) = \{1, 6, 7, 8\}$ and all these extreme nodes have just one neighbor.

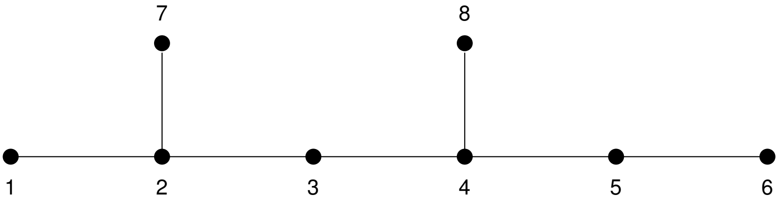
In general, a subgraph of a connected cycle-free graph (N, E) may not be connected,

but for extreme node $h \in S(N, E)$ by definition the subgraph (N_{-h}, E_{-h}) is connected cycle-free graph on N_{-h} with $|N_{-h}| = |N| - 1$ nodes

and moreover the set N_{-h} is the unique satellite of node h in (N, E).

Example of direct calculation of $c_k(N, E)$ for connected cycle-free graph

Consider node 4 in the subgraph on $N' = \{3, 4, 5, 6, 8\}$ of the graph



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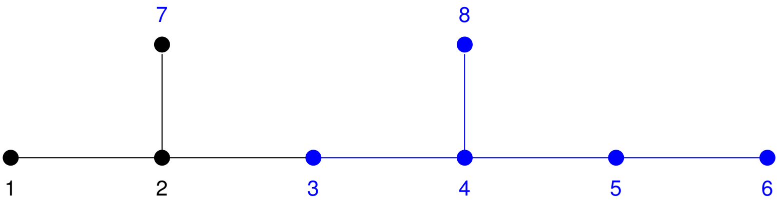


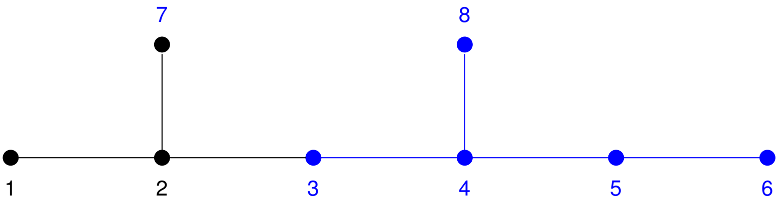
For any linear ordering π feasible with respect to node 4 in $(N', E|_{N'})$, $\pi_1 = 4$ and there are 12 feasible ways to place nodes 3, 5, 6 and 8 after node 4, because the positions of nodes 3, 5 and 8 can be chosen independently from each other, and node 6 has to be chosen after node 5, but not necessarily directly.

$\Longrightarrow \; c_4(N', E|_{N'}) = 12.$

For any linear ordering π feasible with respect to node 3 in $(N', E|_{N'})$, $\pi_1 = 3$, $\pi_2 = 4$, nodes 5 and 8 can be chosen independently from each other, and node 6 has to be chosen after node 5, but not necessarily directly.

$\Longrightarrow \; c_3(N', E|_{N'}) = 3.$

Consider node 4 in the subgraph on $N' = \{3, 4, 5, 6, 8\}$ of the graph



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For any linear ordering π feasible with respect to node 3 in $(N', E|_{N'})$, $\pi_1 = 3$, $\pi_2 = 4$, nodes 5 and 8 can be chosen independently from each other, and node 6 has to be chosen after node 5, but not necessarily directly.

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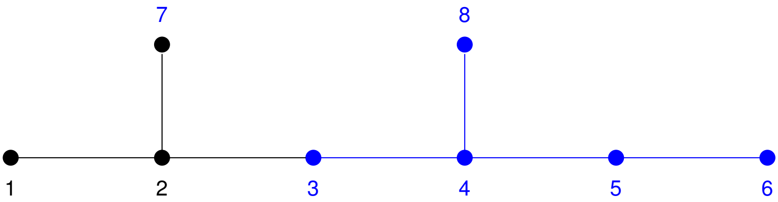
Example of direct calculation of $c_k(N, E)$ for connected cycle-free graph

Consider node 4 in the subgraph on $N' = \{3, 4, 5, 6, 8\}$ of the graph



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$\Longrightarrow \; c_3(N', E|_{N'}) = 3.$

Pascal graph numbers on connected cycle-free graphs

For positive integers n_h, $h = 1, \ldots, k$, with sum equal to n, the *multinomial coefficient* $\binom{n}{n_1, \ldots, n_k}$ is given by

$$\binom{n}{n_1, \ldots, n_k} = \frac{n!}{\prod_{h=1}^{k} n_h!}.$$

For a connected cycle-free graph (N, E) and $k \in N$, $\sum_{h \in B_k^E} |N_{kh}^E| = |N| - 1 \Longrightarrow$ the multinomial coefficient $\binom{|N| - 1}{|N_{kh}^E|,\ h \in B_k^E} = \dfrac{(|N| - 1)!}{\prod_{h \in B_k^E} |N_{kh}^E|!}$ is well defined.

Theorem

For any connected cycle-free graph (N, E) it holds that for every $k \in N$

$$c_k(N,E) = \begin{cases} 1, & |N| = 1, \\ \binom{|N|-1}{|N^E_{kh}|,\ h \in B^E_k} \prod_{h \in B^E_k} c_h(N^E_{kh}, E_{kh}), & |N| \geq 2. \end{cases}$$

In a connected cycle-free graph the Pascal graph number of a node is equal to the multinomial coefficient of the sizes of all its satellites times the product of the Pascal graph numbers of each of its neighbors in the subgraph on the satellite containing this neighbor.

In case of a line-graph all these multinomials are binomials, because for every node there are (at most) two satellites, and moreover, for any node the satellite determined by any of its neighbors is a line-graph and therefore the Pascal graph number of each neighbor in the subgraph on the satellite containing this neighbor is equal to 1 which yields precisely

$$C^k_n = \frac{n!}{(n-k)!k!}.$$

Corollary

If $k \in N$ is an extreme node of a connected cycle-free graph (N, E) and $\{k, h\} \in E$, then $c_k(N, E) = c_h(N_{-k}, E_{-k})$.

Indeed, when k is an extreme node, then for his unique neighbor h, $N^E_{kh} = N_{-k}$, and therefore, $|N^E_{kh}| = |N| - 1$ and $\binom{|N|-1}{|N^E_{kh}|,\ h \in B^E_k} = 1$.

The second corollary shows that similar to binomial coefficients the Pascal graph numbers meet the prime number property.

Corollary

If $|N| - 1$ is a prime number, then the Pascal graph number of any node of a connected cycle-free graph (N, E) other than an extreme node of the graph is divisible by this prime. Moreover, the Pascal graph number of any extreme node of this graph is not divisible by this prime.

Pascal graph numbers on connected cycle-free graphs

Corollary

If $k \in N$ is an extreme node of a connected cycle-free graph (N, E) and $\{k, h\} \in E$, then $c_k(N, E) = c_h(N_{-k}, E_{-k})$.

Indeed, when k is an extreme node, then for his unique neighbor h, $N^E_{kh} = N_{-k}$, and therefore, $|N^E_{kh}| = |N| - 1$ and $\binom{|N|-1}{|N^E_{kh}|,\ h \in B^E_k} = 1$.

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Pascal graph numbers on connected cycle-free graphs

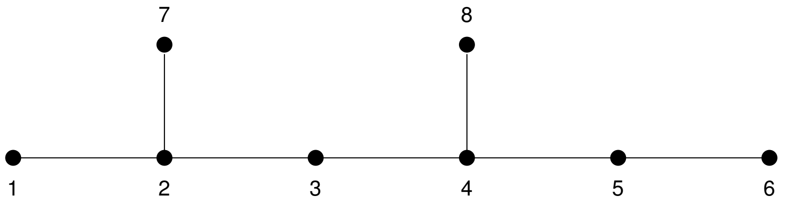
Theorem

For any connected cycle-free graph (N, E) it holds that for every $k \in N$

$$c_k(N,E) = \begin{cases} 1, & |N| = 1, \\ \binom{|N|-1}{|N^E_{kh}|,\ h \in B^E_k} \prod_{h \in B^E_k} c_h(N^E_{kh}, E_{kh}), & |N| \geq 2. \end{cases}$$

Theorem shows that the Pascal graph number of a node can be calculated from the Pascal graph numbers of the neighboring nodes in the smaller subgraphs of the satellites, and therefore provides an iterative procedure to find the Pascal graph numbers for connected cycle-free graphs.

For small enough subgraphs the number of feasible linear orderings is easy to compute, in particular it holds that eventually all satellites become line-graphs, on which the Pascal graph numbers are binomial coefficients.



$$c_2(N,E) = \frac{7!}{1!\,1!\,5!}\; c_1(\{1\}, E|_{\{1\}}) \cdot c_7(\{7\}, E|_{\{7\}}) \cdot c_3(\{3,4,5,6,8\}, E_{\{3,4,5,6,8\}}).$$

Clearly, $c_1(\{1\}, E|_{\{1\}}) = c_7(\{7\}, E|_{\{7\}}) = 1$, and by Corollary 1, $c_3(\{3,4,5,6,8\}, E|_{\{3,4,5,6,8\}}) = c_4(\{4,5,6,8\}, E|_{\{4,5,6,8\}}) = 3$, because the subgraph on $\{4,5,6,8\}$ is a line-graph. Hence,

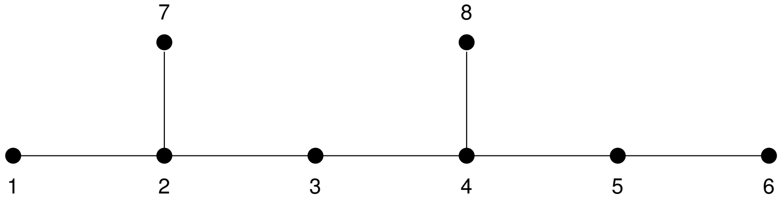
$$c_2(N,E) = \frac{7!}{1!\,1!\,5!} \cdot 1 \cdot 1 \cdot 3 = 42 \cdot 3 = 126.$$

$$c_4(N,E) = \frac{7!}{4!\,2!\,1!} \cdot c_3(\{1,2,3,7\}, E|_{\{1,2,3,7\}}) \cdot c_5(\{5,6\}, E|_{\{5,6\}}) \cdot c_8(\{8\}, E|_{\{8\}}).$$

Clearly, $c_8(\{8\}, E|_{\{8\}}) = c_5(\{5,6\}, E|_{\{5,6\}}) = 1$, and, again by Corollary 1, $c_3(\{1,2,3,7\}, E|_{\{1,2,3,7\}}) = c_2(\{1,2,7\}, E|_{\{1,2,7\}}) = 2$. Hence,

$$c_4(N,E) = \frac{7!}{4!\,2!\,1!} \cdot 2 \cdot 1 \cdot 1 = 210.$$

Pascal graph numbers on connected cycle-free graphs



$$c_2(N,E) = \frac{7!}{1!\,1!\,5!}\, c_1(\{1\}, E|_{\{1\}}) \cdot c_7(\{7\}, E|_{\{7\}}) \cdot c_3(\{3,4,5,6,8\}, E_{\{3,4,5,6,8\}}).$$

Clearly, $c_1(\{1\}, E|_{\{1\}}) = c_7(\{7\}, E|_{\{7\}}) = 1$, and by Corollary 1, $c_3(\{3,4,5,6,8\}, E|_{\{3,4,5,6,8\}}) = c_4(\{4,5,6,8\}, E|_{\{4,5,6,8\}}) = 3$, because the subgraph on $\{4,5,6,8\}$ is a line-graph. Hence,

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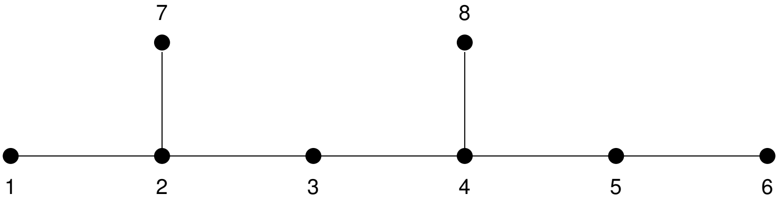
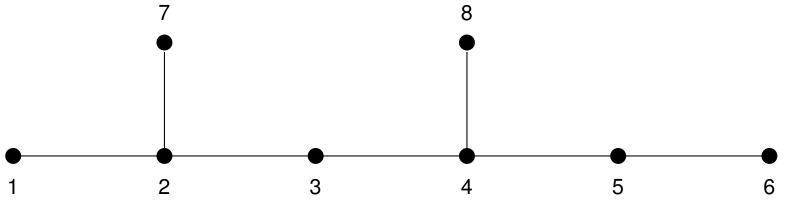


According to Corollary 2, both $c_2(N, E) = 126$ and $c_4(N, E) = 210$ are divisible by the prime number $|N| - 1 = 7$.

For the extreme node 1,

$$c_1(N, E) = c_2(N_{-1}, E_{-1}) = \frac{6!}{1!5!} c_7(\{7\}, E|_{\{7\}}) \cdot c_3(\{3, 4, 5, 6, 8\}, E|_{\{3,4,5,6,8\}}) = 6 \cdot 1 \cdot 3 = 18,$$

which according to Corollary 2 is not divisible by 7.



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which according to Corollary 2 is not divisible by 7.

Let (N, E) be the star graph given by $N = \{0, \ldots, n\}$ and $E = \{\{0, h\} \mid h = 1, \ldots, n\}$, in which each node $k \neq 0$ is connected to the hub at node 0.

By Theorem,

$$c_0(N, E) = n!,$$

because $|N| - 1 = n$, $B_0^E = \{1, \ldots, n\}$, $c_h(N_{0h}^E, E_{0h}) = 1$ for all $h = 1, \ldots, n$ since $N_{0h}^E = \{h\}$.

Each node h, $h = 1, \ldots, n$, is an extreme node connected to only node 0 and the subgraph on its unique satellite $N_{h0}^E = N_{-h}$ is also a star graph with hub node 0, but having in total n nodes.

$\Longrightarrow$ by Corollary 1 for all $h = 1, \ldots, n$,

$$c_h(N, E) = c_0(N_{-h}, E_{-h}) = (n-1)!.$$

Note that $c_0(N, E) = nc_h(N, E)$ for all $h \in N_{-0}$.

$\Longrightarrow$ in a star graph the Pascal graph number of the hub is equal to the sum of the Pascal graph numbers of all other nodes.

Next, let (N, E) be a generalized star graph given by $N = \{0, \ldots, n\}$ with the hub at node 0 having as neighbors nodes $m_1, \ldots, m_k$, that is the graph (N, E) for which for every $h = 1, \ldots, k$ the subgraph on the satellite $N^E_{0m_h}$ of node 0 is a line-graph with n_h nodes having node m_h as an extreme node.

$\Longrightarrow c_{m_h}(N^E_{0m_h}, E_{0m_h}) = 1$ for $h = 1, \ldots, k$ and $\sum_{h=1}^{k} n_h = n$.

$\Longrightarrow$ by Theorem

$$c_0(N, E) = \binom{n}{n_1, \ldots, n_k}.$$

$\Longrightarrow$ in a generalized star graph the Pascal graph number of the hub is equal to the multinomial coefficient for the numbers of nodes in each of the satellites of the hub.

Pascal graph numbers on connected cycle-free graphs

The next result states that for a connected cycle-free graph (N, E) the ratio between the Pascal graph numbers of any two neighbors in the graph is equal to the ratio of the numbers of nodes in the two subgraphs that result from deleting the edge between these two nodes.

Theorem

For any connected cycle-free graph (N, E) *and* $\{k, h\} \in E$ *it holds that*

$$\frac{c_k(N,E)}{c_h(N,E)} = \frac{|N^E_{hk}|}{|N^E_{kh}|}.$$

For the line-graph (N, E) with $N = \{0, \ldots, n\}$ and $E = \{(k, k+1) |\ k = 0, \ldots, n-1\}$ this result reduces to

$$\frac{C^k_n}{C^{k+1}_n} = \frac{k+1}{n-k}.$$

If the Pascal graph number of one node is known, then the Pascal graph numbers of the other nodes can be calculated by successive application of the ratio property.

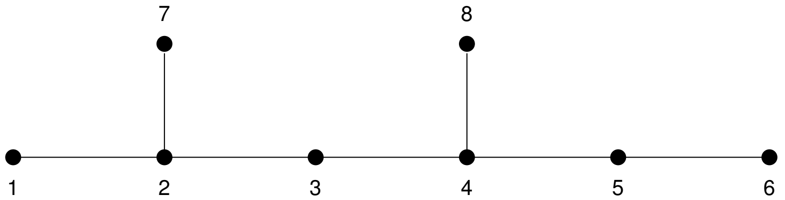
Starting from the node for which the Pascal graph number is known, the Pascal graph numbers of the other nodes follow in any linear ordering which is feasible with respect to the initial node.

The next result immediately follows from Theorem.

Corollary

If in a connected cycle-free graph the deletion of an edge splits the graph in two subgraphs having the same number of nodes, then irrespective to the structure of the two subgraphs obtained, the two nodes adjacent to that edge have equal Pascal graph numbers. Moreover, the Pascal graph number of any other node is smaller.

Note that in Pascal's triangle indeed $C_n^{k-1} = C_n^k$ holds for $k = \frac{1}{2}(n+1)$ when n is odd.



We've found above that $c_4(N, E) = 210$. Since the deletion of the edge $\{3, 4\}$ yields two subgraphs with four nodes in each, due to Corollary 3 we obtain $c_3(N, E) = c_4(N, E) = 210$. Next, by Theorem,

$$c_2(N, E) = \frac{3}{5} c_3(N, E) = 126,$$

which we also know from above. Continuing this way we find

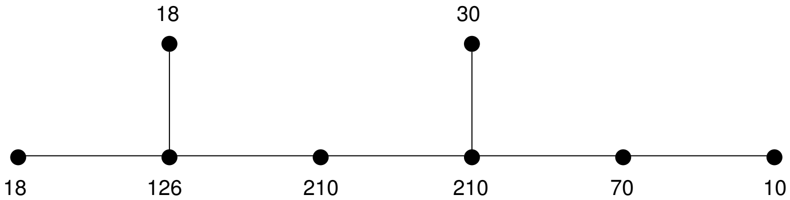
$$c_1(N, E) = c_7(N, E) = \frac{1}{7} c_2(N, E) = 18$$

and

$$c_5(N, E) = \frac{2}{6} c_4(N, E) = 70, \; c_6(N, E) = \frac{1}{7} c_5(N, E) = 10, \; c_8(N, E) = \frac{1}{7} c_4(N, E) = 30.$$

To summarize, the nodes 3 and 4 have equal and maximal Pascal graph numbers and the Pascal graph numbers of the extreme nodes are not divisible by 7, whereas these numbers for all the other nodes are divisible by 7.

Pascal graph numbers on connected cycle-free graphs



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To summarize, the nodes 3 and 4 have equal and maximal Pascal graph numbers and the Pascal graph numbers of the extreme nodes are not divisible by 7, whereas these numbers for all the other nodes are divisible by 7.

Pascal graph numbers on connected cycle-free graphs

The next result generalizes formula

$$c_k(N, E) = c_k(N_{-0}, E_{-0}) + c_k(N_{-n}, E_{-n}).$$

and states that the Pascal graph number of a node in a connected cycle-free graph is equal to the sum of the Pascal graph numbers of that node in all subgraphs obtained by deleting one of the extreme nodes from the graph.

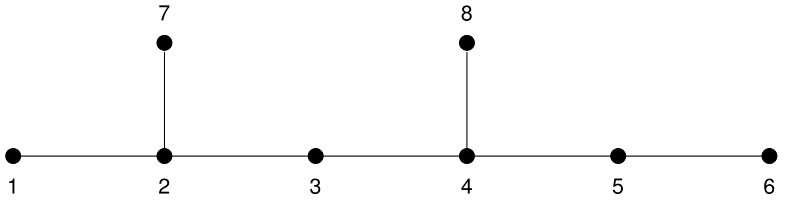
Theorem

For any connected cycle-free graph (N, E), for every $k \in N$,

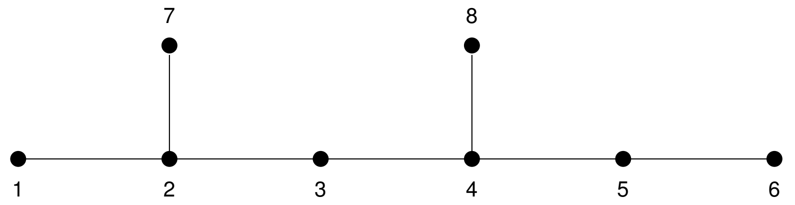
$$c_k(N, E) = \begin{cases} 1, & |N| = 1, \\ \sum_{h \in S(N,E)} c_k(N_{-h}, E_{-h}), & |N| \geq 2, \end{cases}$$

where $c_h(N_{-h}, E_{-h}) = 0$ for all $h \in S(N, E)$.

Pascal graph numbers on connected cycle-free graphs



The graph has four extreme nodes, nodes 1, 6, 7, and 8.



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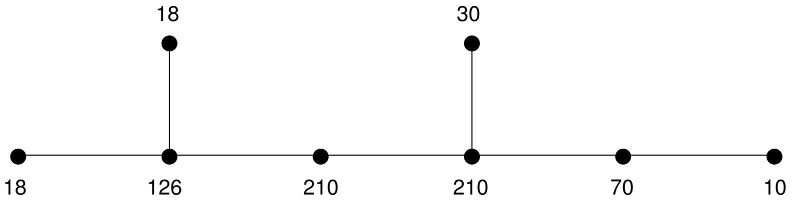

Pascal graph numbers on connected cycle-free graphs

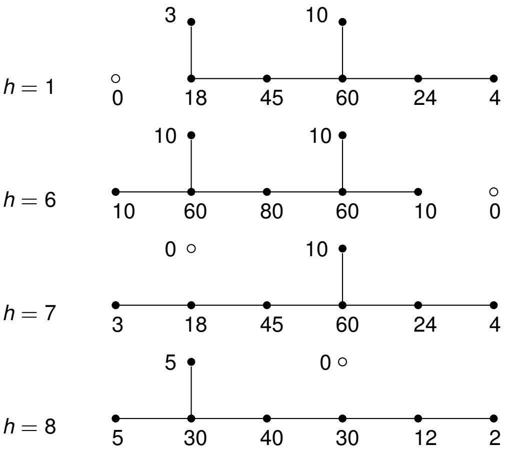

Figure: The Pascal graph numbers for the four subgraphs (N_h, E_h), $h = 1, 6, 7, 8$.

The Pascal graph numbers determine the *stationary distribution of a Markov chain* with the set of nodes as the set of states.

Let (N, E) be a connected cycle-free graph and let s^E be the vector of Pascal graph numbers.

Let P^E be the $|N| \times |N|$ *transition matrix* with the (k, h)th element, $k, h \in N$, given by

$$p^E_{kh} = \begin{cases} \dfrac{|N^E_{kh}|}{|N| - 1}, & \text{if } h \text{ is a neighbor of } k \\ 0, & \text{otherwise,} \end{cases}$$

i.e., when being in node k, the process moves to one of the neighbors with the probabilities proportional to the number of nodes in the satellites.

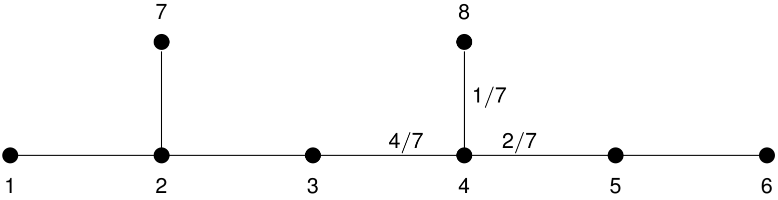

Figure: The transition probabilities from state 4 to the states 3, 5 and 8.

Theorem

For any connected cycle-free graph (N, E) with $|N| \geq 2$ it holds that $s^E P^E = s^E$, i.e., for every $k \in N$ the normalized Pascal graph number $c_k(N, E)/\sum_{h \in N} c_h(N, E)$ is the steady state probability that the Markov chain with transition matrix P^E is in state k.

The steady state probabilities that the Markov chain with transition matrix P^E is in states $k = 3, 4$ are equal to 101/346.

Corollary

For the line-graph on $N = \{0, \ldots, n\}$, for every $k \in N$ the normalized binomial coefficient $C_n^k/2^n$ is the steady state probability that the Markov chain with transition matrix P^E is in state k.

It is well-known that the degrees $d_k(N, E)$, $k \in N$,, when normalized to sum equal to one, are the steady state probabilities of the Markov chain that in any node moves with equal probability to each of its neighbors.

This property also holds for connected graphs that are not cycle-free.

Theorem

For any connected cycle-free graph (N, E) with $|N| \geq 2$ it holds that $s^E P^E = s^E$, i.e., for every $k \in N$ the normalized Pascal graph number $c_k(N, E)/\sum_{h\in N} c_h(N, E)$ is the steady state probability that the Markov chain with transition matrix P^E is in state k.

The steady state probabilities that the Markov chain with transition matrix P^E is in states $k = 3, 4$ are equal to 101/346.

Corollary

For the line-graph on $N = \{0, \ldots, n\}$, for every $k \in N$ the normalized binomial coefficient $C_n^k/2^n$ is the steady state probability that the Markov chain with transition matrix P^E is in state k.

It is well-known that the degrees $d_k(N, E)$, $k \in N$,, when normalized to sum equal to one, are the steady state probabilities of the Markov chain that in any node moves with equal probability to each of its neighbors.

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The Pascal graph numbers and steady state probabilities

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This property also holds for connected graphs that are not cycle-free.

Each linear ordering feasible with respect to some fixed node in a connected graph induces a way to construct the graph by a sequence of increasing connected subgraphs starting from the singleton subgraph determined by this node.

We define the *connectivity centrality measure* as the mapping c on $\mathcal{G}$ that assigns to each connected graph $(N, E) \in \mathcal{G}$ the vector $c(N, E) \in \mathbb{R}^N$ of the Pascal graph numbers of its nodes.

For each node in a given connected graph it measures in how many ways the graph can be generated when starting with this node and adding one by one the other nodes which are connected to at least one node that already has been added.

For a star graph with $n+1$ nodes, the connectivity centrality of the hub is n times as large as the connectivity centrality of each of the n extreme nodes and is therefore equal to the sum of the connectivity centrality of all other nodes.

A well-known centrality measure is the degree measure which assigns to any graph the vector of degrees of its nodes.

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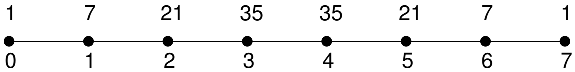

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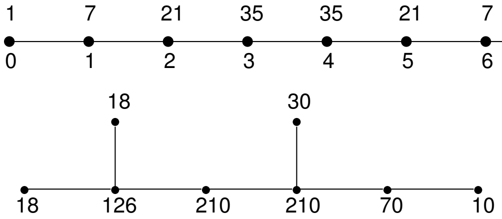

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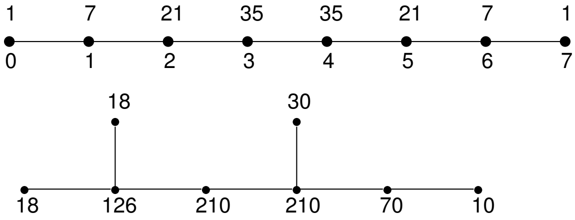

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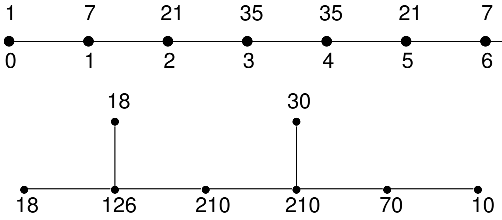

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A centrality measure is a function f which assigns to each connected graph (N, E) a vector $f(N, E) \in \mathbb{R}^N$ with entries $f_i(N, E), i \in N$.

The entry $f_i(N, E)$ measures the centrality of node i in graph (N, E).

We show that the connectivity centrality measure on the subclass of cycle-free connected graphs can be characterized by the three following properties.

Single node normalization A centrality measure f satisfies single node normalization if $f_k(N, E) = 1$ when $N = \{k\}$.

Ratio property A centrality measure f satisfies the ratio property if for every $(N, E) \in \widehat{\mathcal{G}}$ and edge $\{k, h\} \in E$ it holds that $\dfrac{f_k(N, E)}{f_h(N, E)} = \dfrac{|N^E_{hk}|}{|N^E_{kh}|}$.

Extreme node consistency A centrality measure f satisfies extreme node consistency if for every $(N, E) \in \widehat{\mathcal{G}}$ with $|N| \geq 2$ and extreme node $k \in S(N, E)$ it holds that $f_k(N, E) = f_h(N_{-k}, E_{-k})$, where h is the unique neighbor of node k in (N, E).

Theorem

A centrality measure f on the class of cycle-free connected graphs satisfies single node normalization, the ratio property, and extreme node consistency if and only if it is the connectivity centrality measure.

Pascal graph numbers as centrality measure

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For a connected graph (N, E) and subset $N' \subseteq N$ with $E' = E|_{N'}$, N'/E' is the collection of maximal connected subsets of N' in (N, E), called components of N' in (N, E).

For a cycle-free graph (N, E) and node $k \in N$, the components of N_{-k} are the satellites of node k in (N, E).

For an arbitrary connected graph (N, E), $k \in N$ and $C \in N_{-k}/E_{-k}$, the extended subgraph of (N, E) on C with respect to node k is the graph (C, E^k_C) on C with

$$E^k_C = E|_C \cup \{\{i, j\} \subseteq C \mid i \neq j,\ \{i, k\} \in E \text{ and } \{j, k\} \in E\}.$$

So, when two different nodes i and j in C do not form an edge in (N, E) but both form an edge with node k, then edge $\{i, j\}$ is added to the subgraph $(C, E|_C)$.

Theorem

For any connected graph (N, E) it holds that for every $k \in N$

$$c_k(N, E) = \begin{cases} 1, & |N| = 1, \\ \binom{|N|-1}{|C|,\ C \in N_{-k}/E_{-k}} \prod_{C \in N_{-k}/E_{-k}} \sum_{h \in B^E_k \cap C} c_h(C, E^k_C), & |N| \geq 2. \end{cases}$$

Pascal graph numbers on arbitrary connected graphs

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If $k \in N$ is an extreme node of (N, E) and therefore N_{-k}/E_{-k} contains only N_{-k} as its unique element, the expression for $c_k(N, E)$ reduces to generalization of Corollary 1.

Corollary

If k is an extreme node of a connected graph (N, E) with $|N| \geq 2$, then

$$c_k(N, E) = \sum_{h \in B_k^E} c_h(N_{-k}, E^k_{N_{-k}}).$$

In case $k \in N$ is not an extreme node of (N, E), node k is an extreme node of the set $C_{+k} = C \cup \{k\}$ for any component C of N_{-k} in (N, E).

Corollary

If k is not an extreme node of a connected graph (N, E) with $|N| \geq 2$, then

$$c_k(N, E) = \binom{|N| - 1}{|C|,\ C \in N_{-k}/E_{-k}} \prod_{C \in N_{-k}/E_{-k}} c_k(C_{+k}, E|_{C_{+k}}).$$

The latter expression can also be used to express the Pascal graph number of a node that is not an extreme node of a cycle-free connected graph. In that case a satellite C of k in (N, E) is equal to N^E_{kh} with $h \in B^E_k$ being the unique node in C connected to node k, i.e., $C_{+k} = N^E_{kh} \cup \{k\}$, and therefore $c_k(C_{+k}, E|_{C_{+k}}) = c_h(N^E_{kh}, E_{kh})$.

From the last corollary it follows that the first part of Corollary 2 still holds. When $|N| - 1$ is a prime number, then the Pascal graph number of any node that is not an extreme node of a connected graph (N, E) on N is divisible by this prime.

In case the graph contains cycles, however, it might be that the Pascal graph number of an extreme node is divisible by this prime.

For example, if (N, E) is the complete graph, then every node is an extreme node and its Pascal graph number is equal to $(|N| - 1)!$.

Pascal graph numbers on arbitrary connected graphs

When (N, E) is cycle-free, then for any edge $\{k, h\} \in E$ the graph $(N, E \setminus \{\{k, h\}\})$ consists of the two components N^E_{hk} and N^E_{kh} and the ratio property applies.

When (N, E) contains cycles, the ratio property still holds for any edge $\{k, h\} \in E$ which is a bridge in (N, E), i.e., deleting the edge $\{k, h\}$ from E splits the remaining graph in two disconnected subgraphs, (N^E_{kh}, E_{kh}) containing h as a node and (N^E_{hk}, E_{hk}) containing k as a node.

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For any connected graph (N, E) and bridge $\{k, h\} \in E$, it holds that

$$\frac{c_k(N, E)}{c_h(N, E)} = \frac{|N^E_{hk}|}{|N^E_{kh}|}.$$

Note that in a cycle-free connected graph every edge is a bridge. If the graph (N, E) contains cycles and the edge $\{k, h\} \in E$ is not a bridge, then the graph $(N, E \setminus \{\{k, h\}\})$ is still connected and the ratio property does not apply.

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Since the ratio property may not hold in case of graphs with cycles, the results concerning Markov chains and the connectivity centrality measure cannot be generalized to the class of connected graphs.

Finally, the decomposition theorem holds for any connected graph. The proof goes along the same lines of the proof for cycle-free case, because for any linear ordering $\pi \in \Pi(N)$ feasible with respect to a node $k \in N$ in a connected graph (N, E) with $|N| \geq 2$ it holds that $\pi_{|N|}$ is an extreme node of (N, E).

Theorem

For any connected graph (N, E) it holds that for every $k \in N$

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where $c_h(N_{-h}, E_{-h}) = 0$ for all $h \in S(N, E)$.

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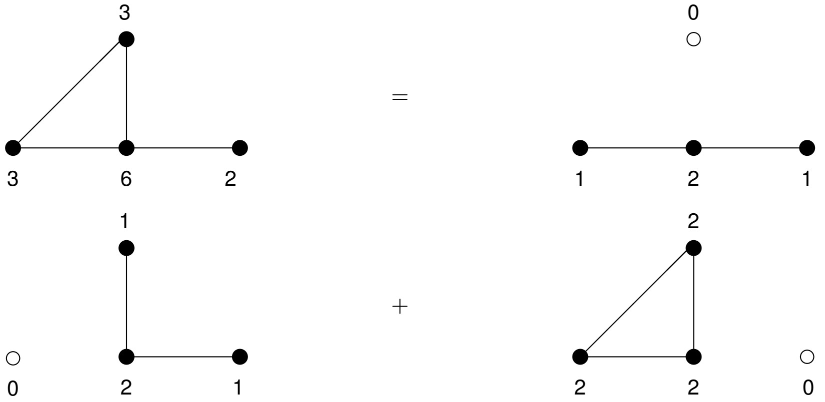

Thank You!