Comparative statics in common value auctions and beyond

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Abstract

The research obtains comparative statics of auction outcome with respect to the distribution of bidders types. The variation of distribution is modelled by different notions of stochastic orders, both univariate (independent types) and multivariate (interdependent types). The main results hold for generalised auction model, which also includes such cases as war of attrition, escalation of conflict and all-pay auction.

Keywords. Stochastic dominance, auctions, common value, comparative statics, war of attrition, multivariate stochastic orders.

JEL Classification Numbers: C72, D44, D82.

1 Introduction

Classical auction theory, originating from Vickrey's [7] and Milgrom and Weber's [4] works, spent a lot of effort comparing different bidding procedures and made great advances in this area. However very few papers consider comparative statics of auction outcome with respect to joint distribution of bidder types. We argue that this question is both very natural and very important. There're many situations in which the aggregate characteristics of participants are changing much quicker then the rules of the game: sometimes changes in legislation are required and sometimes you are just modeling animal competition. Even in some of the most standard economic environments firms often go for different markets, locations, products, promotion strategies — all of it has impact on buyer preferences manifested in joint distribution of their types.

Some papers already tried to address this issue, most notably [8] and [9], however, only for the case of private values. Our research goes far beyond their results in many ways: first, in classical private values setting we obtain results for mean preserving spreads (second order stochastic dominance), second, we consider a generalised auction model that

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includes all the classical auctions as special cases (as well as the famous war of attrition), third, we obtain reasonable results for interdependent types.

Our approach is very different from Milgrom and Weber's information disclosure model. We are doing comparative statics with respect to several multivariate stochastic orders without making any assumptions on original distribution of players types, thus our approach doesn't require affiliated signals. The technique of mathematical proofs that we rely on is also very different from theirs.

The main contribution of our paper are theorems 2, 3 and 4. The first one deals with mean-preserving spreads, which, in general, can have mixed effects; however we can obtain clear comparative statics for very small and very large number of bidders. The second one establishes comparative statics for the diagonal positive dependence order, which is most relevant for game-theoretic competitions. It turns out that zero-measure subset of player types support — the diagonal — has the decisive effect on game outcome. For that reason classical stochastic orders are not "sharp" enough — either they are too weak and fail to restrict the values on the diagonal or they are too strong with "global" point-wise conditions. Finally, our third result provides comparative statics with respect to positive likelihood ratio dependence stochastic order, which has some applications in statistics and can be interpreted as an increase of affiliation between players' signals.

2 To do

1. Write down lemma for n players. Check marginals.
2. Go for expected payment instead of expected bid in special cases.
3. Check expected utilities for general case. Mixed results expected.
4. Anything more about private values?
5. Mention non-affiliation existence papers.

3 Classical setting

Consider a standard n-bidder auction that satisfies all the assumptions of revenue-equivalence theorem. Let $F(x)$ be a cdf of bidder's type. In this case an expected payment of a type x bidder equals:

$$\int_0^x yg(y)dy$$

where $g(y) = (n-1)F^{n-2}(y)f(y)$. To obtain expected payment of bidder we need to integrate this value with respect to distribution of bidder's type:

$$\int_0^b \int_0^x y(n-1)F^{n-2}(y)f(y)dyf(x)dx$$

By changing the variables we obtain:

$$\int_0^b \int_0^x y(n-1)F^{n-2}(y)dF(y)dF(x) = (n-1)\int_0^1 \int_0^x y^{n-2}F^{-1}(y)dydx$$

By changing the order of integration we obtain:

$$(n-1)\int_0^1 \int_0^x y^{n-2}F^{-1}(y)dydx = (n-1)\int_0^1 (1-y)y^{n-2}F^{-1}(y)dy$$

So, the increment of J with respect to F^{-1} can be represented as:

$$\Delta J[u] = (n-1)\int_0^1 (1-x)x^{n-2}\delta(x)dx.$$

Theorem 1 *The increment of bidder's expected payment depends only on variation in (inverse of) bidder's valuation cdf, but not on the cdf itself (al long as monotonicity is respected).*

The monotonicity requirement in the theorem above is crucial, since after we apply a variation to a cdf it should remain a proper distribution function. Mathematically it can be expressed as follows:

$$\left(F^{-1}(x) + \delta(x)\right)' \geq 0.$$

Statement 1 *[2, p. 112] Let θ and θ' be two random variables with cdfs F and F'. Suppose that $F^{-1}(\cdot)$ and $(F')^{-1}(\cdot)$ are well-defined. Then*

$$\begin{cases} \int_0^p F^{-1}(y)dy \geq \int_0^p (F')^{-1}(y)dy, & \forall p \in [0,1), \\ \int_0^1 F^{-1}(y)dy = \int_0^1 (F')^{-1}(y)dy. \end{cases}$$

if and only if θ' is a mean-preserving spread of θ.

Recall that $(F')^{-1}(x) = (F)^{-1}(x) + \delta(x)$ then denote by v the integral of δ from 0 to x:

$$\dot{v} = \delta, \quad v(0) = 0,\ v(1) = 0, \quad v(x) \leq 0,$$

and apply integration by parts:

$$\int_0^1 (1-x)x^{n-2}\delta(x)dx = (1-x)x^{n-2}v(x)\Big|_{x=0}^{x=1} - \int_0^1 (-x^{n-2} + (n-2)(1-x)x^{n-3})v(x)dx$$

Rewriting:

$$-(-x^{n-2} + (n-2)(1-x)x^{n-3}) = (n-2)x^{n-3} - (n-1)x^{n-2} = x^{n-3}((n-1)x - (n-2)).$$

Note that $x^{n-3}(x - \frac{n-2}{n-1})$ is bounded both in x and n. By comparing the integrand with 0 we obtain:

$$\begin{cases} (n-1)x - (n-2) < 0, & x < \frac{n-2}{n-1}; \\ (n-1)x - (n-2) \geq 0 & x \geq \frac{n-2}{n-1}. \end{cases}$$

Observe that if n is large enough, then $J'[u]$ is negative on almost whole $[0,1]$ interval, for the exception of its very end — $(\frac{n-2}{n-1}, 1)$.

$$\Delta J[u] = (n-1)^2 \int_0^1 (x - \frac{n-2}{n-1})x^{n-3}v(x)dx.$$

Theorem 2 *If number of bidders is 2, then every mean-preserving spread of players' valuation distribution results in lower expected payment by a bidder. However, if both old and new distributions have the same support and the number of bidders is large enough then the expected payment starts increasing.*

Remark 1 *If a seller would like to increase his revenue with a mean preserving spread, he should make sure, that the shift actually spreads out the distributions of low-type bidders, i.e. ones with types on the* $[0, \frac{n-2}{n-1}]$ *interval, and leaves unchanged the distribution of high types. For the mean-preserving squeeze the recommendation is reverse.*

Remark 2 *If* $n = 3$ *then the spread symmetric around* $1/2$ *probability (such that* $v(x+1/2) = v(1/2 - x)$, $x \in [0, 1/2]$*) leaves expected payment unchanged, spread affecting interval* $[1/2, 1]$ *more (i.e. that* $v(x + 1/2) > v(1/2 - x)$, $x \in [0, 1/2]$*) decreases the expected payment and spread affecting interval* $[0, 1/2]$ *more (i.e. that* $v(x+1/2) < v(1/2-x)$, $x \in [0, 1/2]$*) increases the expected payment.*

4 Expected utilities

Under the assumptions of a revenue-equivalence theorem an expected utility of a type x bidder equals:

$$xG(x) - \int_0^x yg(y)dy$$

where $G(x) = F^{n-1}(x)$ and $g(y) = (n-1)F^{n-2}(y)f(y)$. To obtain player's expected utility we need to integrate this value with respect to distribution of bidder's type:

$$\int_0^b \left[xF^{n-1}(x) - \int_0^x y(n-1)F^{n-2}(y)f(y)dy \right] f(x)dx$$

By changing the order of integration in the second part and renaming variables we obtain:

$$\int_0^b \left[F^{n-1}(y) - (n-1)(1-F(y))F^{n-2}(y) \right] yf(y)dy$$

and now by changing variables:

$$\int_0^1 \left[x^{n-1} - (n-1)(1-x)x^{n-2} \right] F^{-1}(x)dx.$$

Again, the expected utility is linear in the distribution inverse. So we can do the same trick as above.

$$\Delta J[u] = \int_0^1 \left[x^{n-1} - (n-1)(1-x)x^{n-2} \right] \delta(x)dx.$$

$$\Delta J[u] = -\int_0^1 \left[x^{n-1} - (n-1)(1-x)x^{n-2} \right]' v(x)dx.$$

$$\left[x^{n-1}-(n-1)(1-x)x^{n-2}\right]' = \left[x^{n-1}-(n-1)x^{n-2}+(n-1)x^{n-1}\right]' =$$
$$= \left[nx^{n-1}-(n-1)x^{n-2}\right]' = (n-1)x^{n-3}(nx-(n-2)) = n(n-1)x^{n-3}(x-\frac{n-2}{n}).$$

Thus we obtain the formula:

$$\Delta J[u] = -n(n-1)\int_0^1 (x-\frac{n-2}{n})x^{n-3}v(x)dx.$$

Therefore the same type of comparative statics as above applies in this case.

Remark 3 *Note that conditions for expected payments (i.e. auctioneers payoff) and bidders' expected utilities are different, so it may be the case that the same mean-preserving spread is beneficial both to seller and buyers.*

5 Interdependent types

Now let's consider a generalised auction model. Suppose that there're two ex-ante symmetric players, each one is making his bid b_i. The one making the highest bid is a winner, the other one is a loser. Let θ_1, θ_2 be players' types and b_1, b_2 — their bids. Suppose that player i is the winner, i.e. $b_i > b_j$; he obtains the following utility:

$$U_w^i = B_w(\theta_i, \theta_j, b_j) - b_i C_w(\theta_i).$$

The other bidder's (loser's) utility equals:

$$U_l^j = B_l(\theta_j, \theta_i, b_j) - b_j C_l(\theta_j).$$

The functions above should be smooth in all variables. Note that this utility functions are quasi-linear in player's own bid. This assumption can be relaxed[1] to separability in own bid and other player's type/bid (i.e. second term may be non-linear), however we can go no further without losing the "auction-style" mechanics — for example, Tullock contest with incomplete information isn't covered by our setup. But a lot of other important models are included, for example:

1. First price auction: $C_w = 1$, $B_w = \theta_i$ $C_l = B_l = 0$,
2. Second price auction: $C_w = 0$, $B_w = \theta_i - b_j$ $C_l = B_l = 0$,
3. War of attrition: $C_w = 0$, $B_w = \theta_i - b_j$ $C_l = -1$, $B_l = 0$,
4. All-pay auction: $C_w = 1$, $B_w = \theta_i$ $C_l = -1$, $B_l = 0$.

Let $F(\theta_1, \theta_2)$ be joint distribution of player types. We will assume that it has density with $[0,1]\times[0,1]$ support. Throughout this section it would be in the focus of our interest —recall that the main question of the paper is: "How does the expected bid change when F changes". To address it we will first find the expression for the expected bid and then introduce the necessary multivariate stochastic orders.

The following natural assumptions will help us later:

[1] As we conjecture.

1. $C_w \geq 0$, $C_l \geq 0$ — players prefer to make lower bids if the result doesn't change.

We will proceed with standard heuristic for symmetric equilibrium in auction models: suppose that $b(\cdot)$ is a continuous and monotonic equilibrium strategy; then we can write down player's interim expected utility. Let θ be player's true type and $\bar{\theta}$ is the type he pretends to be.

$$U(\bar{\theta}, \theta) = \int_0^{\bar{\theta}} B_w(\theta, \tilde{\theta}, b(\tilde{\theta}))dF(\tilde{\theta}|\theta) - b(\bar{\theta})C_w(\theta)F(\bar{\theta}|\theta) + \\ + \int_{\bar{\theta}}^1 B_l(\theta, \tilde{\theta}, b(\tilde{\theta}))dF(\tilde{\theta}|\theta) - b(\bar{\theta})C_l(\theta)(1 - F(\bar{\theta}|\theta)).$$

We can write down the FOC and then simplify them:

$$\frac{\partial}{\partial \bar{\theta}} U(\bar{\theta}, \theta) = B_w(\theta, \bar{\theta}, b(\bar{\theta}))f(\bar{\theta}|\theta) - b'(\bar{\theta})C_w(\theta)F(\bar{\theta}|\theta) - b(\bar{\theta})C_w(\theta)f(\bar{\theta}|\theta) - \\ - B_l(\theta, \bar{\theta}, b(\bar{\theta}))f(\bar{\theta}|\theta) - b'(\bar{\theta})C_l(\theta)(1 - F(\bar{\theta}|\theta)) + b(\bar{\theta})C_l(\theta)f(\bar{\theta}|\theta) = 0.$$

In equilibrium the players are willing to report their true type, thus we obtain:

$$b'(\theta)[C_w(\theta)F(\theta|\theta) + C_l(\theta)(1 - F(\theta|\theta))] = \\ = [B_w(\theta, \theta, b(\theta)) - B_l(\theta, \theta, b(\theta)) - b(\theta)(C_w(\theta) - C_l(\theta))]f(\theta|\theta).$$

Let's use the following notation to simplify the expression above:

$$g(\theta, F) = \frac{f(\theta|\theta)}{C_w(\theta)F(\theta|\theta) + C_l(\theta)(1 - F(\theta|\theta))} = \frac{f(\theta, \theta)}{C_w(\theta)\int_0^\theta f(x, \theta)dx + C_l(\theta)\int_\theta^1 f(x, \theta)dx},$$

$$L(b, \theta) = B_w(\theta, \theta, b(\theta)) - B_l(\theta, \theta, b(\theta)) - b(\theta)(C_w(\theta) - C_l(\theta)).$$

So the differential equation for equilibrium strategy can be rewritten the following way:

$$b'(\theta) = L(b, \theta)g(\theta, F).$$

Note that its only $g(\theta, F)$ that depends on initial distribution of types. The player's expected bid can be easily found given his strategy ($F(\theta)$ is a marginal of $F(\theta_1, \theta_2)$:

$$J = \int_0^1 b(\theta)dF(\theta)$$

Assumption 1 *From now on we will assume that the heuristic above "works" and that $b(\cdot)$ is indeed smooth and increasing equilibrium strategy.*

We make this assumption mainly because this heuristic indeed works in all the applications of interest ([4],[5]) and cause our goal in not to investigate all the numerous special models but to derive universal comparative static results.

Let's assume that we add some small variance to the distribution of players' types ΔF. That results in some small variation Δg of $g(\theta, F)$. Now we are going to investigate how value of J changes with Δg. To do it we will linearise J with respect to Δg with the help of some standard optimal control theory technique [3].

$$\begin{cases} J = \int_0^1 b(\theta) dF(\theta), \\ b'(\theta) = L(b, \theta) g, \\ b(0) = 0. \end{cases}$$

Let's write down the Hamilton-Pontryagin function:

$$H = bf + \psi L(b, \theta) g$$

The conjugate system

$$\begin{cases} b' = L(b, \theta) g, \\ \psi' = -\frac{\partial}{\partial b} H = -f(\theta) - g \frac{\partial}{\partial b} L(b, \theta) \psi, \\ b(0) = 0, \\ \psi(1) = 0. \end{cases}$$

and the Frechet derivative of our functional:

$$\frac{\partial}{\partial g} H = \psi L(b, \theta).$$

We end up with the following system:

$$\begin{cases} \Delta J = \int_0^1 \psi L(b, \theta) \Delta g d\theta, \\ b'(\theta) = L(b, \theta) g, \\ \psi' = -\frac{\partial}{\partial b} H = -f(\theta) - g \frac{\partial}{\partial b} L(b, \theta) \psi, \\ b(0) = 0, \\ \psi(1) = 0. \end{cases}$$

Note that it follows from the system that $\psi(\theta) \geq 0$ if $\theta \leq 1$ (since $f(\theta) > 0$). $L(b, \theta)$ stands for difference in utilities from winning and losing if both players have the same types and both are playing equilibrium strategy. It also can't be negative since $b(\theta)$ is increasing by our assumption and $g(\theta, F)$ is always nonnegative.

So, the the only thing that matters for our comparative statics is the sign of Δg. Let's recall its formula:

$$g(\theta, F) = \frac{f(\theta, \theta)}{C_w(\theta) \int_0^\theta f(x, \theta) dx + C_l(\theta) \int_\theta^1 f(x, \theta) dx},$$

we see that it is affected by the joint distribution primarily through its behavior along the diagonal. However standard notions of multivariate stochastic orders or positive dependence orders do not impose conditions on the diagonal density of joint distributions[2], hence these notions are likely to be either too weak or too strong for our purposes.

[2]The diagonal of the square has zero measure, so "normal" stochastic orders have very good reason to ignore it.

We will proceed in two steps, at first we will slightly modify stochastic order notion from [6] to apply it here, secondly we will make use of classical positive dependence order that relies on the well-known notion of affiliated random variables [4].

Definition. Let $\theta = (\theta_1, \theta_2)$ and $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2)$ be two pairs of symmetric random variables with joint densities $f(x, y)$ and $\hat{f}(x, y)$ respectively and same support $[0, 1] \times [0, 1]$. $(\hat{\theta}_1, \hat{\theta}_2)$ are more *diagonally positive dependent*[3] then (θ_1, θ_2) if two following conditions hold:

1. $\forall \alpha, \beta > 0 \ \forall y \in [0, 1]$ we have

$$\frac{\hat{f}(y, y)}{\alpha \int_0^y \hat{f}(z, y)dz + \beta \int_y^1 \hat{f}(z, y)dz} \geq \frac{f(y, y)}{\alpha \int_0^y f(z, y)dz + \beta \int_y^1 f(z, y)dz}.$$

2. $f(x, y)$ and $\hat{f}(x, y)$ have the same marginals.

The second condition just says that two distributions should belong to the same Frechet class, which is standard for positive dependence orders. The first one says that density on the diagonal increases relative to any positive combination of densities on the line below this point and above this point.

It's also worth stressing that this notion is indeed *diagonal*, as it puts almost no restrictions on the behavior of θ and $\hat{\theta}$ in the off-diagonal points, and indeed this was the reason of paper [6] to introduce new notion — even weak standard multivariate positive dependence orders, such as PQD order (see [2, p. 387]), restrict behavior of θ and $\hat{\theta}$ in irrelevant areas of θ and $\hat{\theta}$ support.

The following theorem directly follows from DPD order definitions and the linearization argument above:

Theorem 3 *Let $\theta = (\theta_1, \theta_2)$ and $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2)$ be player's types. Suppose that $\hat{\theta}$ is more diagonally positive dependent than θ. Then expected bid is greater for $\hat{\theta}$ than for θ.*

Clearly in this case the sign of Δg is always positive and we can divide a large variation into sequence os small ones.

To proceed further we need the definition of positive likelihood ratio dependence order.

For any two intervals I_1 and I_2 of the real line, let us denote $I_1 \leq I_2$ if $x_1 \in I_1$ and $x_2 \in I_2$ imply that $x_1 \leq x_2$. Let $\theta = (\theta_1, \theta_2)$ and $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2)$ be two pairs of symmetric random variables with joint CDFs $F(x, y)$ and $\hat{F}(x, y)$ respectively and same support $[0, 1] \times [0, 1]$. Let $F(x, y)$ and $\hat{F}(x, y)$ have the same marginals. Suppose that

$$\frac{F(I_1, J_1)F(I_2, J_2)}{F(I_1, J_2)F(I_2, J_1)} \leq \frac{\hat{F}(I_1, J_1)\hat{F}(I_2, J_2)}{\hat{F}(I_1, J_2)\hat{F}(I_2, J_1)},$$

whenever $I_1 \leq I_2$ and $J_1 \leq J_2$. Here the generic notation F(I, J) is obvious. Then we say that θ is smaller than $\hat{\theta}$ in the PLRD order.

[3] Or greater in diagonal positive dependence order (DPD order)

When $F(x,y)$ and $\hat{F}(x,y)$ have (continuous or discrete) densities $f(x,y)$ and $\hat{f}(x,y)$, then the expression above is equivalent to

$$\frac{f(x_1,y_1)f(x_2,y_2)}{f(x_1,y_2)f(x_2,y_1)} \leq \frac{\hat{f}(x_1,y_1)\hat{f}(x_2,y_2)}{\hat{f}(x_1,y_2)\hat{f}(x_2,y_1)},$$

whenever $x_1 \leq x_2$ and $y_1 \leq y_2$.

If the appropriate derivatives exists the expression above is also equivalent to:

$$\frac{\partial^2}{\partial x \partial y} \ln f(x,y) \leq \frac{\partial^2}{\partial x \partial y} \ln \hat{f}(x,y).$$

Theorem 4 *Let $\theta = (\theta_1, \theta_2)$ and $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2)$ be player's types. Suppose that $\hat{\theta}$ is greater than θ in PLRD order. Then expected bid is greater for $\hat{\theta}$ than for θ.*

The proof of this theorem is rather complicated, so we will proceed in steps. Our main idea would be to construct an auxiliary optimization problem such that its maximum would correspond to situation when PLRD shift cannot decrease player's expected bid. Literally, the idea of the proof is that a function with non-positive derivative on some interval cannot be increasing in any point on it — however, in our case there would be not just one dimension, but many.

First let's take densities f and $\hat{f}$ such that the second one is greater then the first one in PLRD order. Let's denote by δ the following value:

$$\hat{f}(x,y) = e^{\delta(x,y)} f(x,y), \qquad \delta(x,y) = \ln \frac{\hat{f}(x,y)}{f(x,y)}.$$

Now let's divide $[0,1]$ interval into n equal parts and consider discrete approximation of our density: now f_{ij} equals to the density mass in square with coordinates (i,j). For other functions adding index (C_w^i, for example) means taking its value in the middle of i-th interval. Due to assumptions made above all the necessary convergence results can be obtained by a straightforward limit argument. In order to avoid additional symmetry constraints from now on we will consider only the upper triangle of support: $j \geq i$. Thus, if $\hat{f}$ dominates f we have the following restrictions on δ:

$$\begin{cases} \delta_{ii} + \delta_{jj} \geq 2\delta_{ij}, & 1 \leq i < j \leq n, \\ \sum\limits_{i \leq j} f_{ij} + \sum\limits_{i > j} f_{ji} = \sum\limits_{i \leq j} f_{ij} e^{\delta_{ij}} + \sum\limits_{i > j} f_{ji} e^{\delta_{ji}}, & j = 1..n. \end{cases}$$

The first condition stands for PLRD dominance — all other constraints in definition follow from these ones. The second one requires marginals to remain the same.

Let $J[\delta]$ be the functional of our interest (for example — expected payment in auction). For a second, suppose the contrary to our theorem's claim — the value of J can be decreased by switching from f to $\hat{f}$, despite the PLRD. Then $J[\delta]$ cannot have minimum in δ at f, so the FOC should be violated for some sufficiently large n. We are going to write conditions on when it is not the case. We have the following constrained optimization problem:

$$\begin{cases} J[\delta] \to \min, & \\ 2\delta_{ij} - \delta_{ii} - \delta_{jj} \le 0, & 1 \le i < j \le n, \\ \sum\limits_{i \le j} f_{ij} + \sum\limits_{i>j} f_{ji} - \sum\limits_{i \le j} f_{ij} e^{\delta_{ij}} - \sum\limits_{i>j} f_{ji} e^{\delta_{ji}} = 0, & j = 1..n. \end{cases}$$

To write down the first order conditions let's take small δ so that we linearize everything with respect to it; this way the equality constraints will be:

$$\sum_{i \le j} \delta_{ij} f_{ij} + \sum_{i>j} \delta_{ji} f_{ji} + \bar{o}(||\delta||) = 0, \quad j = 1..n.$$

Let's use the following notation:

$$c_{ij} = \lim_{\delta \to 0} \frac{\partial}{\partial \delta_{ij}} J[\delta].$$

Here ψ_{ij} are the dual variables for the inequality constraints, and ϕ_j — for the equalities.

$$\begin{cases} c_{ii} = -\sum\limits_{j<i} \psi_{ji} - \sum\limits_{j>j} \psi_{ij} + \phi_i f_{ii}, \\ c_{ij} = 2\psi_{ij} + f_{ij}(\phi_i + \phi_j), \\ \psi_{ij} \le 0. \end{cases} \tag{1}$$

Denote by

$$\Psi_i = \sum_{j<i} \psi_{ji} + \sum_{j>j} \psi_{ij},$$

so we have

$$\phi_i = \frac{1}{f_{ii}}(c_{ii} + \Psi_i).$$

By substituting the expression above back into system we obtain:

$$c_{ij} = 2\psi_{ij} + \frac{f_{ij}}{f_{ii}}(c_{ii} + \Psi_i) + \frac{f_{ij}}{f_{jj}}(c_{jj} + \Psi_j),$$

and thus

$$c_{ij} - \frac{f_{ij}}{f_{ii}} c_{ii} - \frac{f_{ij}}{f_{jj}} c_{jj} = 2\psi_{ij} + \frac{f_{ij}}{f_{ii}} \Psi_i + \frac{f_{ij}}{f_{jj}} \Psi_j, \qquad 1 \le i < j \le n.$$

The only question that remains is whether this linear system has a non-positive solution. This system has nonnegative matrix, we claim that it is non-degenerate as well. To check it let's go back to the original linear constraint system and show that it is non-degenerate. Just recall that in our case the number of constraints exactly equals the number of variables and square matrix transpose preserves non-degeneracy.

So we need to show that the following system has only trivial solution:

$$\begin{cases} 2\delta_{ij} = \delta_{ii} - \delta_{jj}, & 1 \le i < j \le n, \\ \sum\limits_{i \le j} \delta_{ij} f_{ij} + \sum\limits_{i > j} \delta_{ji} f_{ji} = 0, & j = 1..n. \end{cases}$$

Indeed, by substituting the first group of constraints into the second we obtain:

$$\sum_{i \le j} (\delta_{ii} + \delta_{jj}) f_{ij} + 2\delta_{jj} f_{jj} + \sum_{i > j} (\delta_{ii} + \delta_{jj}) f_{ji} = 0, \quad j = 1..n.$$

This smaller system has diagonal dominance and therefore is non-degenerate.

Let's go back to line of the proof: we've just shown that the following system has non-negative non-degenerate matrix. Therefore if its left hand side is negative, the solution is negative as well.

$$c_{ij} - \frac{f_{ij}}{f_{ii}} c_{ii} - \frac{f_{ij}}{f_{jj}} c_{jj} = 2\psi_{ij} + \frac{f_{ij}}{f_{ii}} \Psi_i + \frac{f_{ij}}{f_{jj}} \Psi_j, \qquad 1 \le i < j \le n.$$

So we can state the following lemma:

Lemma 1 *If for all sufficiently large n, $c_{ij} - \frac{f_{ij}}{f_{ii}} c_{ii} - \frac{f_{ij}}{f_{jj}} c_{jj} \le 0$ for $1 \le i < j \le n$ then the value of $J[\delta]$ can't be decreased with all the PLRD constraints satisfied.*

Now we are almost done — the only thing that remains is to apply the lemma above to our problem. Let's recall the definition of the $J[\delta]$ in our case:

$$\begin{cases} \Delta J = \int_0^1 \psi L(b, \theta) \Delta g d\theta), \\ b'(\theta) = L(b, \theta) g, \\ \psi' = -\frac{\partial}{\partial b} H = -f(\theta) - g \frac{\partial}{\partial b} L(b, \theta) \psi, \\ b(0) = 0, \\ \psi(1) = 0. \end{cases}$$

Note that only Δg depends on δ, so we can rewrite it the following way:

$$\Delta J = \int_0^1 \omega(\theta) \Delta g d\theta, \qquad \omega(\theta) \ge 0.$$

Now let's recall that

$$\Delta g(\theta, F) = \frac{f(\theta, \theta) e^{\delta(\theta, \theta)}}{C_w(\theta) \int_0^\theta f(x, \theta) e^{\delta(x, \theta)} dx + C_l(\theta) \int_\theta^1 f(x, \theta) e^{\delta(x, \theta)} dx} - \\ - \frac{f(\theta, \theta)}{C_w(\theta) \int_0^\theta f(x, \theta) dx + C_l(\theta) \int_\theta^1 f(x, \theta) dx},$$

Now we can write down a discrete approximation of ΔJ:

$$J \approx \sum_{i=1}^{n} \frac{\omega_i f_{ii}}{C_w^i \sum_{j=1}^{i-1} f_{ji} + C_l^i \sum_{j=i+1}^{n} f_{ij}}$$

$$\Delta J \approx \sum_{i=1}^{n} \frac{\omega_i f_{ii} \delta_{ii}}{C_w^i \sum_{j=1}^{i-1} f_{ji} + C_l^i \sum_{j=i+1}^{n} f_{ij}} - \\ - \sum_{i=1}^{n} \frac{\omega_i f_{ii}}{(C_w^i \sum_{j=1}^{i-1} f_{ji} + C_l^i \sum_{j=i+1}^{n} f_{ij})^2} \left(C_w^i \sum_{j=1}^{i-1} f_{ji} \delta_{ji} + C_l^i \sum_{j=i+1}^{n} f_{ij} \delta_{ij} \right).$$

One final step left: we need to show that

$$c_{ij} - \frac{f_{ij}}{f_{ii}} c_{ii} - \frac{f_{ij}}{f_{jj}} c_{jj} \leq 0$$

But note, that c_{ij} always comes from the right part of the expression and is negative while c_{ii} comes from the left and is positive. Thus we are done.

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