A Comparison of Stochastic and Fuzzy Orderings

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Applications of Histograms Comparison

There are many applications where the comparison of random variables (probability distributions) is required w.r.t. relationship of type "more-less":

- theory of reliability and risk
  [Stochastic Orders in Reliability and Risk. LNS, 208, Springer,2013]
- biomedical research [eg, Kottas A., 2011]
- comparison of experimental results [eg, Shnoll S.E.]
- comparison of indicators functioning of homogeneous (organizational, technical, etc.) systems
- economics [eg, Shorrocks A.F.: Ranking Income Distributions, 1983]
- models of social welfare
  [eg, Measurement of Inequality: Rothschild M., Stiglitz J., Sen A.]
- simulation of fuzzy preferences [eg, Fodor & Roubens, 1994]
- decision making under uncertainty [eg,Vanegas & Labib 2001]
  etc.

Objects of Comparison

- stochastic orderings
  [Shaked M., Shanthikumar J.G.: Stochastic Orders. Springer, 2007]
- fuzzy orderings
  [eg, Wang X., Ruan D., Kerre E.E.: Mathematics of Fuzziness – Basic Issues. Springer, 2009]
- fuzzy random orderings
  [eg, Aiche F., Dubois D.: An Extension of Stochastic Dominance to Fuzzy Random Variables. LNCS, 6178, Springer, 2010, pp.159–168; Couso I., Dubois D. A perspective on the extension of stochastic orderings to fuzzy random variables, IFSA-EUSFLAT, 2015, Gijon, Spain]

Outline of Presentation

1. Stochastic and Fuzzy Orderings
   - examples of pairwise comparison
   - relationships between comparisons
2. A Comparison of Histograms by the Method of Minimal Transformations
   - transportation metric
   - examples of comparison of histograms on base of calculation of their minimal transformation
3. Summary and Conclusion

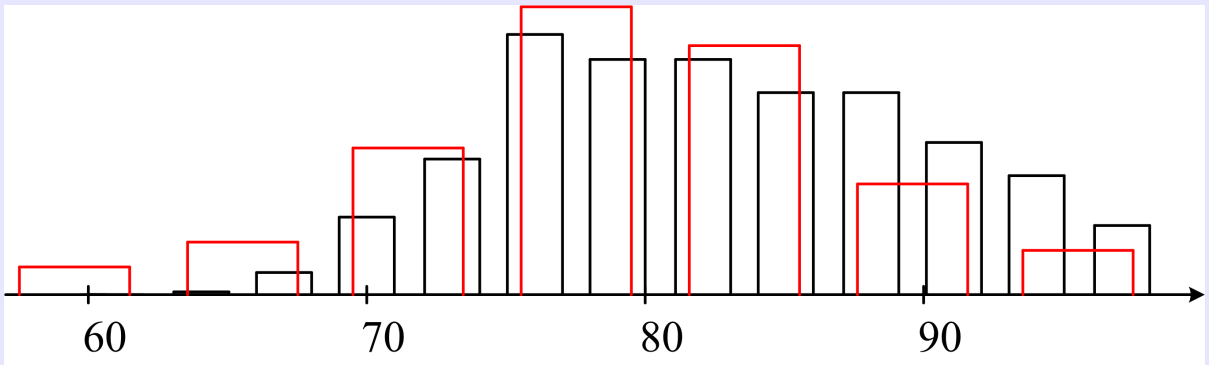
Problem Statement of Comparison of Histograms

Let $\mathcal{U} = \{U\}$ be a set of all histograms of the form $U = (x_i, u_i)_{i \in I}$, $x_i < x_{i+1}$, $i \in I$.

Examples:

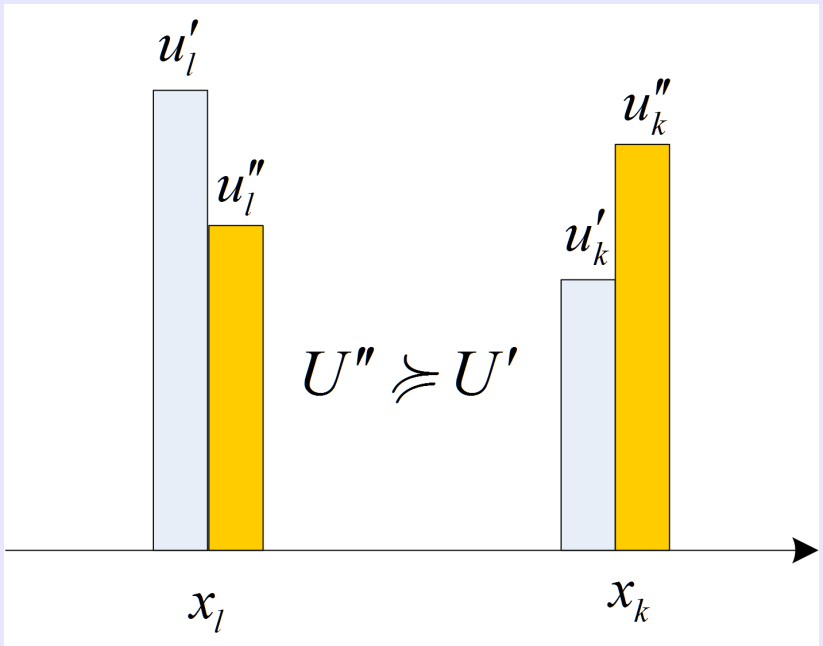
1. $u_i = P\{U = x_i\}$, $i \in I$ – discrete random variable (RV) U
2. $u_i = \mu_U(x_i)$, $i \in I$ – fuzzy variable (FV) U, where μ_U is a membership function on the universal set $(x_i)_{i \in I}$

We want define the total preorder relation R (reflexive, complete and transitive relation) on $\mathcal{U}$: $(U, V) \in R \Leftrightarrow U \succeq V$.



Ordering Arguments of Histograms

The relation R must be coordinated with the condition of ordering histogram arguments in accordance with an increase in their importance:
if $U' = (x_i, u'_i)$, $U'' = (x_i, u''_i)$ be two histograms for which $u'_i = u''_i$ for all $i \neq k, l$ and $u'_l - u''_l = u''_k - u'_k \geq 0$ then $U'' \succeq U'$ for $k > l$ and $U' \succeq U''$ for $k < l$.



Stochastic Orderings

1. Dominance in Expectation:

$$U \succcurlyeq_{me} V, \text{ if } E[U] \geq E[V].$$

In general $U \succcurlyeq_{mef} V$, if $E[f(U)] \geq E[f(V)]$, where f is a some utility function.

2. Stochastic dominance:

$$U \succcurlyeq_{sd} V, \quad \text{if} \quad F_U(t) \leq F_V(t) \quad \text{for all} \quad t \in \mathbb{R},$$

where $F_U(t) = \sum_{i:x_i<t} u_i$ be a distribution function of RV U.

Disadvantages: 1) it may be determined not on all set of $\mathcal{U}^2$;
2) it is very sensitive to distortions.

Since $E[U] - E[V] = \int_{-\infty}^{\infty} (F_V(t) - F_U(t))\, dt$ is true, then

$$U \succcurlyeq_{sd} V \Rightarrow U \succcurlyeq_{me} V.$$

3. Stochastic precedence

$U \succcurlyeq_{sp} V$, if $P\{U \geq V\} \geq P\{U \leq V\} \Leftrightarrow P\{U \geq V\} \geq \frac{1}{2}(1 + P\{U = V\})$

If RVs $U = (x_i, u_i)_{i \in I}$ and $V = (y_j, v_j)_{j \in J}$ are independent then

$$P\{U \geq V\} = \sum_{(i,j):\, x_i \geq y_j} u_i v_j = \int_{-\infty}^{\infty} F_V(t) dF_U(t) \geq \int_{-\infty}^{\infty} F_U(t) dF_U(t) = \frac{1}{2},$$

if $U \succcurlyeq_{sd} V$. Therefore

$$U \succcurlyeq_{sd} V \Rightarrow U \succcurlyeq_{sp} V.$$

Disadvantage: in general the stochastic precedence is not a transitive relation.

The Relationship between the Different Dominances

Theorem

The following relations hold:

1. $U \succcurlyeq_{sd} V \Rightarrow U \succcurlyeq_{me} V$;
2. $U \succcurlyeq_{sd} V \Rightarrow U \succcurlyeq_{sp} V$, *if* U, V *are independent RVs.*

The stochastic precedence does not follow from stochastic dominance in the general case of dependent RVs. For example, if $U \sim N(m_U, \sigma_U^2)$ and $V \sim N(m_V, \sigma_V^2)$ then

$$U \succcurlyeq_{sd} V \Leftrightarrow m_U \geq m_V \text{ and } \sigma_U^2 = \sigma_V^2,$$

but

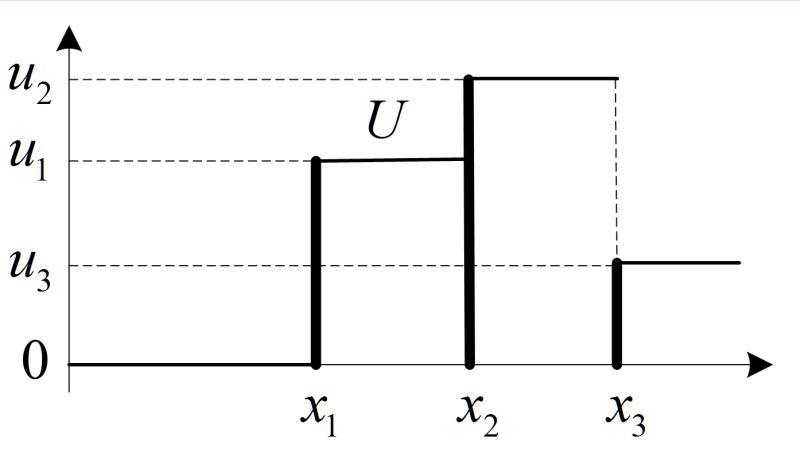
$$U \succcurlyeq_{sp} V \Leftrightarrow m_U \geq m_V.$$

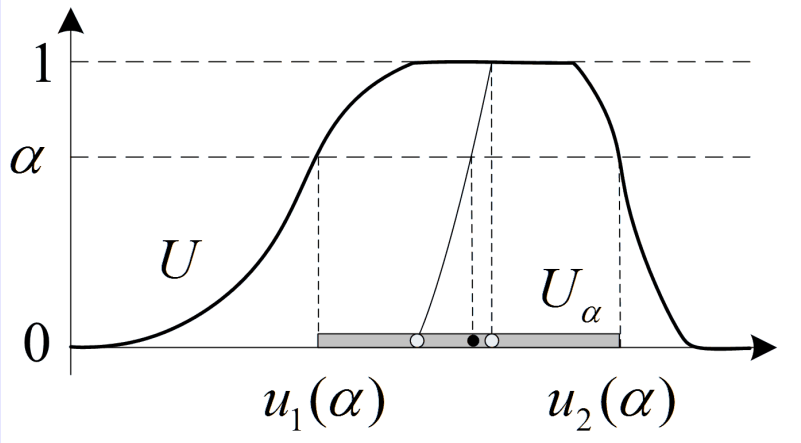
Fuzzy Sets and Fuzzy Numbers

The fuzzy set (FS) U with membership function $u_i = \mu_U(x_i)$ defined on the set $X = (x_i)_i \subseteq \mathbb{R}$ can be associated with histogram $U = (x_i, u_i)_i$.

The FS U with a step membership function μ_U corresponds to the histogram $U = (x_i, u_i)_i$, where $\mu_U(t) = u_i$ for $x_i < t \le x_{i+1}$, $i \in I$ and $\mu_U(t) = 0$ for $t \notin (\min x_i, \max x_i]$.

The FS $U = (u_i)_i$ is a fuzzy number (FN), if α-cuts $U_\alpha = \{t : \mu_U(t) \ge \alpha\}$ are non empty convex closed sets $\forall \alpha \in (0, 1]$ and $\max_i u_i = 1$.





Comparison with the Help of the Ranking Function

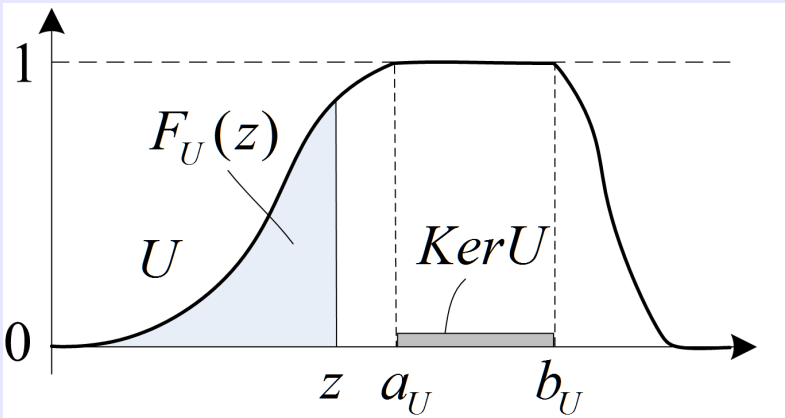
A certain function (index, defuzzification operation) $F : \mathcal{U} \to \mathbb{R}$ is determined and we believe that $U \succcurlyeq V$ $(U \sim V)$, if $F(U) \geq F(V)$ $(F(U) = F(V))$.

Examples:

- Adamo index (1980): $A_\alpha(U) = u_2(\alpha)$, where fixed value $\alpha \in (0, 1]$ characterized the measure of risk of wrong decisions;
- generalized Yager index (1981): $Y_\lambda(U) = \int_0^1 (\lambda u_1(\alpha) + (1 - \lambda)u_2(\alpha))d\alpha$, where coefficient $\lambda \in [0, 1]$ characterized the level of optimism of decision maker;
- centroid index $C(U) = \sum_i x_i u_i / \sum_i u_i$. If U is a RV then $C(U) = E[U]$.

The Generalized Yager Index

Let U be a FN and $F_U(z) = \int_{-\infty}^{z} \mu_U(t)dt$, $|U| = \int_{\mathbb{R}} \mu_U(t)dt$ be a cardinality of the FN, $a_U = \inf\{t : \ \mu_U(t) = 1\}$ and $b_U = \sup\{t : \ \mu_U(t) = 1\}$ are borders of kernel of the FN.



Theorem

If U is a FN and $|U| < \infty$, then $Y_\lambda(U) = a_U + \lambda\,|U| - F_U(a_U)$.

The Relationship between the Different Dominances

Let $U \succcurlyeq_{Y_\lambda} V$, if $Y_\lambda(U) \geq Y_\lambda(V)$.

Corollary

- *If* $|U| = |V| < \infty$, *then*

$$Y_\lambda(U) \geq Y_\lambda(V) \Leftrightarrow a_U - F_U(a_U) \geq a_V - F_V(a_V).$$

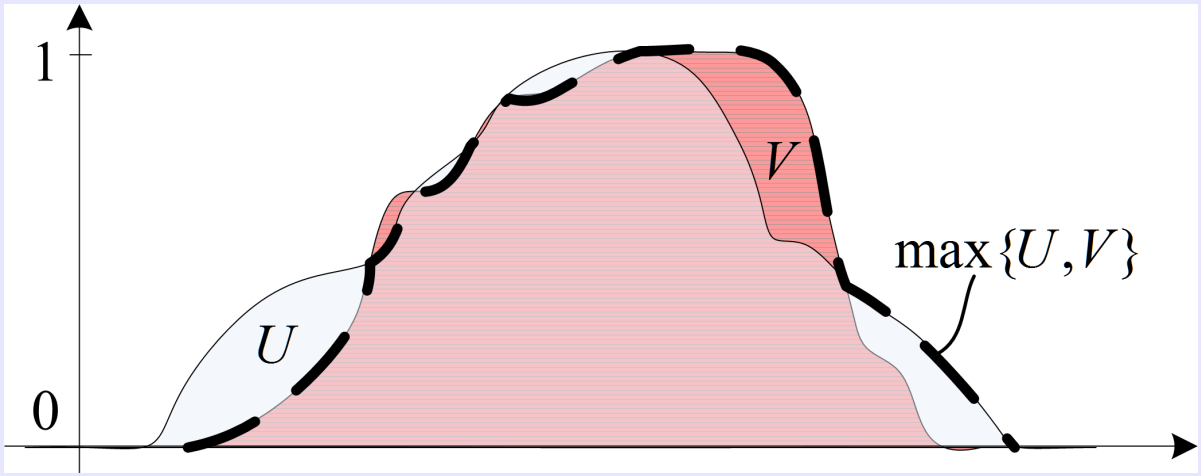
- *If* $|U| = |V| < \infty$ *and* $a_U \geq a_V$, *then*

$$U \succcurlyeq_{sd} V \Rightarrow U \succcurlyeq_{Y_\lambda} V \quad \textit{for every} \quad \lambda \in [0,1].$$

The dominance w.r.t. Yager index in general does not follow from stochastic dominance.

The Comparison Methods Based on the Calculation of Distance to the Reference FN

Kerre index (1982): $K(U) = d_1(U, U_0)$, where FN $U_0 = \max\{U : U \in \mathcal{U}\}$ (it is calculated using the Zadeh's extension principle), d_1 is a Hamming distance. Then $U \succcurlyeq V$, if $K(U) \leq K(V)$.



We can consider the pairwise Kerre index of two FNs U and V: $K(U, V) = d_1(U, \max\{U, V\})$. Then $U \succcurlyeq_K V$, if $K(U, V) \leq K(V, U)$.

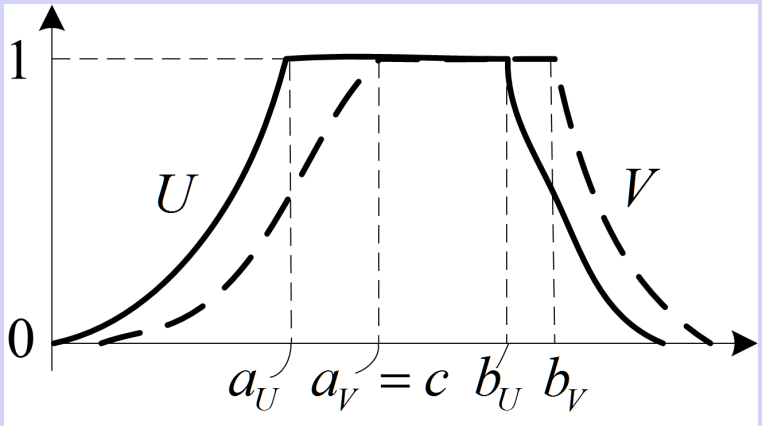
Comparison with the help of a Pairwise Kerre Index

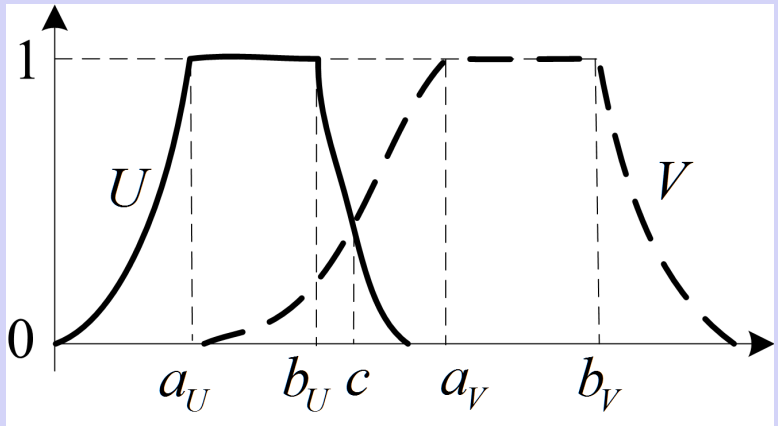
Theorem

If U *and* V *are FNs,* $|U| = |V| < \infty$, *then*

$$K(V,U) - K(U,V) = 2\left(F_V(c) - F_U(c)\right),$$

where $c = \underset{[\min\{b_U,b_V\},\max\{a_U,a_V\}]}{\arg\min} \{\max\{\mu_U(t),\mu_V(t)\}\}$, *if* $\min\{b_U,b_V\} \le \max\{a_U,a_V\}$ *and* $c = \max\{a_U,a_V\}$ *otherwise.*





Corollary

We have under the same conditions:

1. $U \succcurlyeq_{sd} V \Rightarrow U \succcurlyeq_K V$;
2. $U \succcurlyeq_{Y_\lambda} V \Rightarrow U \succcurlyeq_K V$ *for every* $\lambda \in [0,1]$.

The comparison with the help of Yager index and Kerre index does not follow from comparison with the help of mathematical expectation and vice versa.

The Methods Based on the Calculation of the Pairwise Comparison Index

Baas–Kwakernaak index (1977) is an example of this approach:

$$BK(U,V) = \sup_{x_i \geq x_j} \min\{u_i, v_j\}.$$

This index is a "fuzzy" analogue of calculating the probability $P\{U \geq V\}$ for independent RVs U and V, if operations of addition and multiplication replaced on the min and sup respectively.

Let $U \succcurlyeq_{BK} V$, if $BK(U,V) \geq BK(V,U)$. The relation $\succcurlyeq_{BK}$ is transitive in contrast to the "similar" stochastic precedence. The relation $\succcurlyeq_{BK}$ does not follow from $\succcurlyeq_{sp}$ and vice versa.

The pair Kerre index $K(U,V) = d_1(U, \max\{U,V\})$ is another example of a pairwise comparison index which determines the relation $U \succcurlyeq_K V$ if $K(U,V) \leq K(V,U)$. The relation $\succcurlyeq_K$ does not transitive in general.

The Descriptions of Comparisons in Terms of the Distribution Function

Thus we shown that many comparisons can be described in terms of the distribution function of RV or FV:

1. $U \succcurlyeq_{me} V \Leftrightarrow E[U] - E[V] = \int_{-\infty}^{\infty} (F_V(t) - F_U(t))\, dt \geq 0$;
2. $U \succcurlyeq_{sd} V \Leftrightarrow F_V(t) - F_U(t) \geq 0 \quad \forall t$;
3. $U \succcurlyeq_{sp} V \Leftrightarrow \int_{-\infty}^{\infty} F_V(t) dF_U(t) - \int_{-\infty}^{\infty} F_U(t) dF_V(t) \geq 0$, if RVs U and V are independent;
4. $U \succcurlyeq_{Y_\lambda} V \Leftrightarrow a_U - F_U(a_U) \geq a_V - F_V(a_V)$, if $|U| = |V| < \infty$;
5. $U \succcurlyeq_{K} V \Leftrightarrow F_V(c) - F_U(c) \geq 0$, if $|U| = |V| < \infty$.

We can see that the dominance in expectation and the stochastic precedence have an integral character w.r.t. distribution function. But dominances w.r.t. Yager index and index Kerre have a point character w.r.t. distribution function.

Disadvantages of Comparisons

- Kerre index is weakly sensitive when comparing the large number of FNs
- Baas–Kwakernaak index is insensitive when comparing two FNs with intersecting kernels
- unimodality of histograms required for Adamo and Yager indices
- comparison histograms with the help of fuzzy orders requires data normalization as a rule
- comparison histograms with the help of fuzzy orders is often difficult interpret in terms of subject area

Monge–Kantorovich Transportation Problem

G. Mong (1781) and L.V. Kantorovich (1942).

Solution of this problem has led to a notion of Kantorovich–Rubinstein metric (transportation metric) and to some of their variations: Wasserstein metric, Mallows metric, EMD metric (Earth Mover's Distance metric) etc.

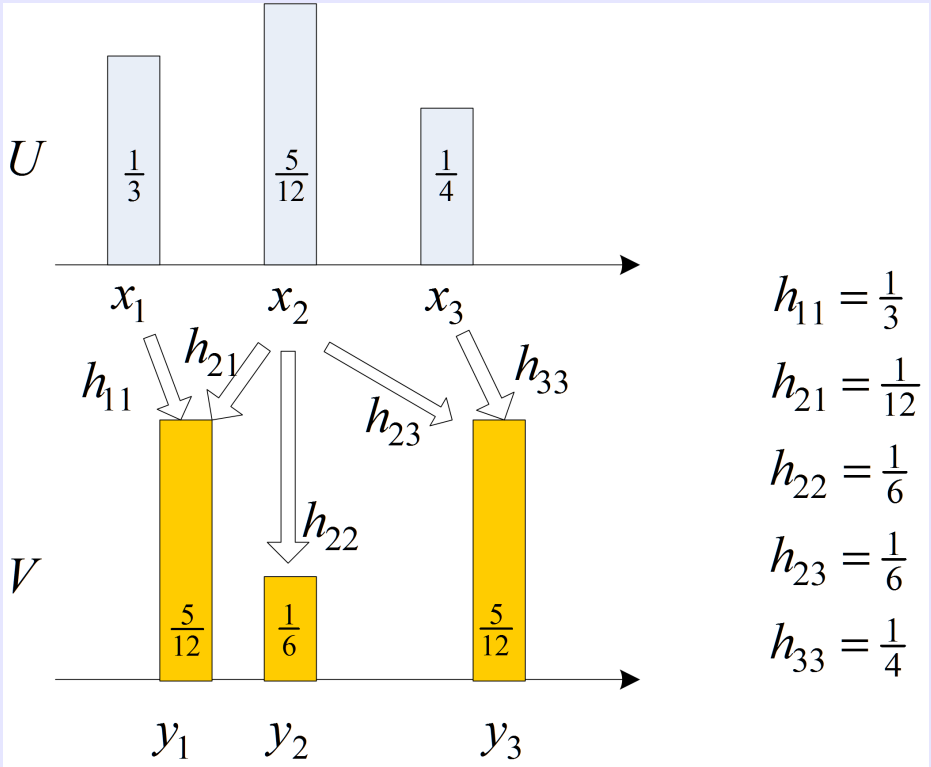
Mallows metric for discrete RVs $U = (x_i, u_i)_{i \in I}$, $V = (y_j, v_j)_{j \in J}$. Let h_{ij} be a flow between the "points" x_i and y_j, that satisfy the conditions (the plan of transportation):

$$h_{ij} \geq 0, \quad \sum_i h_{ij} = v_j \ \forall j, \ \sum_j h_{ij} = u_i \ \ \forall i \tag{1}$$

and $d_{ij} = |x_i - y_j|$ be a distance between the "points" x_i and y_j.

Mallows Metric

Then Mallows metric is equal $d(U,V) = \inf\limits_{(h_{ij})\in\mathcal{H}} \sum\limits_{i,j} h_{ij} d_{ij}$.



$$d(U,V) = \int_0^1 \left|F_U^{-1}(x) - F_V^{-1}(x)\right| dx = \int_{-\infty}^{\infty} \left|F_U(t) - F_V(t)\right| dt.$$

Earth Mover's Distance (EMD) Metric

If $U = (x_i, u_i)_{i \in I}$, $V = (y_j, v_j)_{j \in J}$ are discrete FVs, then the following transportation plan should be performed

$$\tilde{\mathcal{H}}: \quad h_{ij} \geq 0, \quad \sum_i h_{ij} \leq v_j \ \forall j, \quad \sum\nolimits_j h_{ij} \leq u_i \ \ \forall i,$$

$$\sum_i \sum_j h_{ij} = \min \left\{ \sum_i u_i, \sum_j v_j \right\}$$

and EMD metric is equal

$$d(U, V) = \sum_{i,j} h_{ij}^{opt} d_{ij} \Big/ \sum_{i,j} h_{ij}^{opt},$$

where $H^{opt} = (h_{ij}^{opt})$ is an optimal plan of transportation.

Comparison of Histograms on Base of Calculation of their Minimal Transformation

Example 1. Let

$$r_1(U,V) = \sum\nolimits_{i,j} h_{ij}^{opt}(x_i - y_j).$$

Then $U \succcurlyeq_{tr1} V$, if $r_1(U,V) \geq 0$. Note that we have for RVs

$$r_1(U,V) = E[U] - E[V] = \int_0^1 (F_V(x) - F_U(x))\,dx.$$

This comparison for RVs is equivalent to comparison of mathematical expectation: $U \succcurlyeq_{tr1} V \Leftrightarrow U \succcurlyeq_{me} V$.

Example 2. Let

$$r_2(U,V) = \sum\nolimits_{i,j} h_{ij}^{opt}(x_i - y_j)_+,$$

where $(t)_+ = \frac{1}{2}(t + |t|)$. Then $U \succcurlyeq_{tr2} V$, if $r_2(U,V) \geq r_2(V,U)$.

We have

$$r_2(U,V) + r_2(V,U) = d(U,V), \qquad r_1(U,V) = r_2(U,V) - r_2(V,U).$$

Therefore $U \succcurlyeq_{tr2} V \Leftrightarrow U \succcurlyeq_{tr1} V \Leftrightarrow U \succcurlyeq_{me} V$.

The relation $\rho(U,V) = r_2(U,V)/d(U,V)$, if $U \neq V$ and $\rho(U,U) = \frac{1}{2}$ otherwise is a so called **probabilistic relation** (reciprocal relation, ipsodual relation), i.e. [0,1]-valued relation ρ satisfying $\rho(U,V) + \rho(V,U) = 1$.

Example 3. Let

$$r_3(U,V) = \inf_{(h_{ij})\in\mathcal{H}} \sum\nolimits_{i,j} h_{ij}\varphi(x_i - y_j),$$

where nondecreasing function φ satisfies the condition $\varphi(t) = 0$ for $t \le 0$. Then $U \succcurlyeq_{tr3} V$, if $r_3(U,V) \ge r_3(V,U)$. The function φ defines the restrictions on the "long-distance" transportation.

These comparisons have very clear interpretation. The comparison $U \succcurlyeq_{tr} V$ means that the "cost of transportation" from histogram U to V is less than from V to U.

Summary and Conclusion

1. Simple formulas for calculating some comparisons were obtained, as well as established relationships between the various comparisons.
2. These results explain why the fuzzy ranking may be very different from stochastic ranking. The comparisons with the help of Yager index or Kerre index have a point character for distribution functions. But the comparison with the help of expectation has an integral character w.r.t. distribution functions.
3. A new approach to comparison of discrete RVs or FVs with the help of calculation the minimal transformation between two histograms was developed. This method has many advantages:
   - it has a good interpretability for many applied problems;
   - it does not assume the unimodality of the compared histograms as many methods of comparison of FNs.

   At the same time this method is equivalent to the comparison w.r.t. expectation in the simplest case.

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Thanks for you attention

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