Formation of coalition structures as a non-cooperative game

Dmitry Levando, [*]

Abstract

The paper defines a non-cooperative simultaneous finite game to study coalition structure formation. The game definition embeds a coalition formation mechanism, which includes a maximum coalition size, a set of eligible partitions with coalitions sizes no greater than the given number and coalition structure formation rule. The paper defines a family of nested non-cooperative games. The family is parametrized by a size of a maximum coalition size in a coalition structure of every game. Every game in the family has an equilibrium in mixed strategies. The equilibrium encompasses intra and inter group externalities, what makes it different from Shapley value. Presence

[*]This paper comes as a development of the 3-rd and the 4-th chapters of my PhD Thesis "Essays on Trade and Cooperation" at Ca Foscari University, Venezia, Italy. Acknowledge: Nick Baigent, Phillip Bich, Alex Boulatov, Emiliano Catonini, Giulio Codognato, Sergio Currarini, Luca Gelsomini, Izhak Gilboa, Olga Gorelkina, Piero Gottardi, Roman Gorpenko, Eran Hanany, Mark Kelbert, Ludovic Julien, Alex Kokorev, Dmitry Makarov, Francois Maniquet, Igal Miltaich, Stephane Menozzi, Roger Myerson, Ariel Rubinstein, Marina Sandomirskaya, William Thompson, Konstantin Sonin, Simone Tonin, Dimitrios Tsomocos, Eyal Winter, Shmuel Zamir. Special thanks to Fuad Aleskerov, Shlomo Weber and Lev Gelman. Many thanks for advices and for discussion to participants of SIGE-2015 (an earlier version of the title was "A generalized Nash equilibrium"), CORE 50 Conference, CEPET 2016 Workshop, Games 2016 Congress, Games and Applications 2016. All possible mistakes are only mine. This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.
E-mail for correspondence: dlevando (at) hse.ru.

of individual payoff allocation makes it different from a strong Nash, coalition-proof equilibrium, and some other equilibrium concepts. The following papers demonstrate applications of the proposed model.

Keywords: Noncooperative Games
JEL : C72

1 Subject of the paper

The research topic of this paper was inspired by John Nash's "Equilibrium Points in n-person games "(1950). This remarkably short, but highly influential note of only 5 paragraphs established an equilibrium concept and the proof of its existence which did not require an explicit specification of a final coalition structure for a set of players. Prior to Nash's paper, the generalization of the concept of equilibrium provided by von Neumann for the case of two-players zero-sum game was done by portioning the players into two groups and regarding several players as a single player. However up to now, non of these approaches resulted into expected progress in studying intra- and inter group externalities between players.

In this paper "coalition structure", or "partition"[1] for short, as a collection of non-overlapping subsets from a set of players, which in a union make the original set. A group, or a coalition, is an element of a coalition structure or of a partition.[2]

A partition induces two types of effects on a player's payoff. The first, through actions of players of the same coalition. These effects will be addressed as *intra*-coalition externalities. The second, from all other player, who are outside coalition, and can belong to different coalitions. This effects will be addressed as *inter*-group externalities.

[1]Existing literature uses both terms.
[2]the same.

Nash (1953) suggested that cooperation should be studied within a group and in terms of non-cooperative fundamentals. This conjecture is known now as the Nash Program. Cooperative behavior was understood as an activity inside a group. Nash did not write explicitly about multi-coalition framework or coalition structures. Coalition structures allow us to study inter-coalition externalities, along with intra-coalition ones.

The best analogy for the difference between Nash Program and current research is the difference between partial and general equilibrium analysis in economics. The former isolates a market ignoring cross-market interactions, the latter explicitly studies cross market interactions.

The research agenda of the paper is: how to construct coalition structures from actions of self-interested agents. Moreover, the paper offers a generalization of a non-cooperative game from Nash (1950) to address the problem of coalition structure formation absent in Nash (1953). The contributions of this paper are: a construction of a non-cooperative game with an embedding coalition structure formation mechanism, and a parametrization of all constructed games by a number of deviators. The two subsequent papers will demonstrate various applications of the model to different types of games, how to measure stability of appearing coalition structures in terms of a self-interested behavior and applications to network games.

The paper has the following structure: Section 2 presents an example, on why studying inter-coalition externalities requires including coalition structures into individual strategy sets, Section 3 presents a general model of the game, Section 4 presents an example of the game. The last section discusses the approach used and the results of the paper.

2 An example: corporate dinner game

The simple example below shows that to study coalition structures we need to incorporate them into a strategy set, this is different from Hart and Kurz

(1978) say, they study coalition formation in terms of choice of a coalition by every player.

Consider a game of 4 players: A is a President; B is a senior vice-president; C_1, C_2 are two other vice-presidents. A coalition is a group of players at *one* table. Every player may sit only at one table. A coalition structure is an allocation of all players over no more than four tables. Empty tables are not taken into account.

Individual set of strategies is a set of all coalition structures for the players, i.e. a set of all possible allocations of players over 4 non-empty tables. A player chooses one coalition structure. A set of strategies in the game is a direct product of four individual strategy sets. The choice of all players is a point in the set of strategies of the game.

Preferences are such that everyone (besides A) would like to have a dinner with A, but A only with B. Everyone wants players outside his table to eat individually, due to possible dissipation of rumors or information exchange. No one can enforce others to form or not to form coalitions.

In every partition any coalition (i.e. a table) is formed only if everybody at the table agrees to have dinner together, otherwise a player eats alone. The same coalition may belong different coalition structures, but with different allocations of players beyond it, see lines 1 and two in Table 1 further.

The game is simultaneous and one shot. A realization of a final partition (a coalition structure) depends on choices of coalition structures of all players. Example. Let player A choose $\{\{A, B\}, \{C_1\}, \{C_2\}\}$; player B chooses $\{\{A, B\}, \{C_1\}, \{C_2\}\}$; player C_1 choose $\{\{A, C_1\}, \{B\}, \{C_2\}\}$, and player C_2 choose $\{\{A, C_2\}, \{B\}, \{C_2\}\}$. Then the final partition is $\{\{A, B\}, \{C_1\}, \{C_2\}\}$. It is clear that a strong Nash equilibrium (Aumann, 1960), which is based on a deviation of a coalition of any size, does not discriminate between the coalition structures mentioned above.

Players have preferences over coalition structures. Payoff profile of all players in the game should be defined for every final coalition structure. Table

1 presents coalition structures only with the best individual payoffs.[3] Thus only some partitions from the big set of all strategies deserve attention. The first column is a number of a strategy. The second column is an allocation of players over coalition structures, and also is a list of best final coalition structures. The third column is an individual payoff profile of all players for every listed coalition structure. The fourth column is a list of values for coalitions in coalition structure if to calculate values using cooperative game theory.

Table 1: Strategies and payoffs in the corporate dinner game

num	Best final partitions	Non-cooperative payoff profile $(U_A, U_B, U_{C_1}, U_{C_2})$	Values of coalitions as in cooperative game theory
1	$\{\{A,B\},\{C_1\},\{C_2\}\}$	(10,10,3,3)	20_{AB}, 3_{C_1}, 3_{C_2}
2*	$\{\{A,B\},\{C_1,C_2\}\}$	(8,8,5,5)	16_{AB}, $10_{C_1,C_2}$
3	$\{\{A,C_1\},\{B,C_2\}\}$	(3,5,10,5)	13_{AC_1}, 10_{BC_2}
4	$\{\{A,C_1\},\{B\},\{C_2\}\}$	(3,3,10,3)	13_{AC_1}, 3_B, 3_{C_2}
5	$\{\{A,C_2\},\{B,C_1\}\}$	(3,5,5,10)	13_{AC_2}, 10_{BC_1}
6	$\{\{A,C_2\},\{B\},\{C_1\}\}$	(3,3,3,10)	13_{AC_1}, 3_B, 3_{C_1}
7	all other partitions	(0,0,0,0)	all payoffs are 0

The game runs as follows. Players simultaneously announce choices of a desirable coalition structure, then a final coalition structure is formed according to the rule above, and payoffs are assigned.

Players A and B would always like to be together. Being rational they would choose a coalition structure with the highest payoff for them, i.e. the strategy 1. By the same reason the first best choices of C_1 and C_2 would be to choose coalition structures with A. But A will never choose to be with either of them. The unavailability of the first best makes C_1 and C_2 to choose option 2.[4] By doing so they do not disallow a coalition $\{A,B\}$, but reduce

[3]All other coalition structures have significantly lower payoffs.

[4]In sociology this behavior is referred as a cooperation: players C_1 and C_2 group

payoffs for A and B. And players A and B cannot prevent this (or to insure against).

On the other hand, if players A and B choose strategy 2 they will obtain coalition $\{A, B\}$ in any case, but in a different final coalition structure. In terms of mixed strategies this means that an equilibrium mixed strategy for A and for B is a whole probability space over points, two coalition structures 1 and 2.

From the forth column we can see that the corresponding cooperative game has an empty core, strong Nash equilibrium can not be applied here also, as well as coalition-proof equilibrium.

The constructed game has a unique equilibrium. In terms of individual payoffs it is characterized as the second-best efficient for everybody. Equilibrium coalition structure contains two coalitions. This equilibrium is different from the strong Nash and a coalition proof equilibria. It does not require super-additivity of payoffs. The game is based only on the individual unbounded rationality of every player.

There are also differences from partition approach (Yi, 1999), as, for example. there is no initial allocation of players over coalitions.

3 Formal setup of the model

There is a set of players N, with a general element i, $N = \{i\}$, a size of N is a finite integer, $2 \leq \#N < \infty$.

Every game has a parameter K, $K \in \{1, \#N\}$. Take an example to illustrate two interpretations for K. Suppose there are $N = 10$ players in the game, and a maximum coalition size is $K = 5$. Then no more than five players are required to dissolve any coalition of the maximum size five. The reverse is also true: we need no more than 5 players to form any coalition of

together against other options when they are not together and have lower payoffs. This problem will be addressed in the accompanying paper.

size five. Closeness of a game construction requires the two interpretations for K.[5]

Number K induces a family of coalition structures (family of partitions) over N: $\mathcal{P}(K) = \{P\colon P \subset 2^N, \#P \leq K\}$, such that families for different K are nested: $\mathcal{P}(K = 1) \subset \ldots \subset \mathcal{P}(K) \subset \ldots \subset \mathcal{P}(K = N)$. The bigger is K, the more coalition structures (or partitions) are involved. Thus a set of values for K induces nested families of coalition structures.

For every family of coalition structures $\mathcal{P}(K)$ a player has a finite strategy set $S_i(K) = \{S_i(P_i)\colon P_i \in \mathcal{P}(K)\}$, P_i is a partition from $\mathcal{P}(K)$, which i may choose.[6] Individual strategy sets for different K are nested:

$$S_i(K = 1) \subset \ldots \subset S_i(K) \subset \ldots \subset S_i(K = N).$$

The set game strategies with a fixed K is $S(K) = \times_{i \in N} S_i(K)$, a direct product of individual strategy sets of all players. However a realization of a final partition for player i may differ from one's choice P_i. It is clear that an increase in K induces nested strategy sets: $S(K = 1) \subset \ldots \subset S(K = N)$.

In the example above we have seen that we need some mechanism $\mathcal{R}(K)$, which assigns to every strategy profile $s \in S(K)$ a final coalition structure P, $P \in \mathcal{P}(K)$.

Definition 1. ***A coalition structure formation mechanism*** $\mathcal{R}(K)$ *is a family of mappings such that:*

1. *Domain of* $\mathcal{R}(K)$ *is a set of all strategy profiles of* $S(K)$.

2. *A range of* $\mathcal{R}(K)$ *is a finite set of subsets of* $S(K)$*, where every strategy profile* $s = (s_1, \ldots, s_N)$ *from* $S(K)$*, is assigned only one destination partition,* P*. Number of subsets of* $S(K)$ *is equal to a cardinality of* $\mathcal{P}(K)$*.*

[5] If we change value of K we change the game.

[6] Finite strategies are chosen are used in Nash (1950),

3. *$\mathcal{R}(K)$ shares $S(K)$ into coalition structure specific strategy sets, $S(K) = \cup_{P\in\mathcal{P}(K)} S(P)$, where $S(P)$ is a coalition structure specific strategy set.*

   .

4. *Two different coalition structures, $\bar{P}$ and $\tilde{P}$, $\bar{P} \neq \tilde{P}$, have different coalition structure strategy sets $S(\bar{P}) \cap S(\tilde{P}) = \emptyset$.*

Formally the same:

$$\mathcal{R}(K)\colon S(K) = \times_{i\in N} S_i(K) \mapsto\colon \begin{cases} \forall s = (s_1, \ldots, s_N) \in S(K)\ \exists P \in \mathcal{P}(K)\colon s \in S(P), \\ S(K) = \cup_{P\in\mathcal{P}(K)} S(P), \\ \forall \bar{P}, \tilde{P} \in \mathcal{P}(K), \bar{P} \neq \tilde{P} \Rightarrow S(\bar{P}) \cap S(\tilde{P}) = \emptyset. \end{cases}.$$

Hence there are two ways to construct $S(K)$: in terms of initial individual strategies $S(K) = \times_i S_i(K)$, or in terms of specific realized partition strategies $S(K) = \cup_{P\in\mathcal{P}(K)} S(P)$. Representation of $S(K)$ in terms of coalition structure specific strategy sets may not be a direct product of sets, see an example in the next section.

An increase in K holds consistency of coalition structure formation mechanisms for different K: $\mathcal{R}(K = 1) \subset \ldots \subset \mathcal{R}(K) \subset \ldots \subset \mathcal{R}(K = N)$. Thus coalition structure formation mechanisms for different values of the parameter K are nested.

Payoffs in the game are defined as state-contingent payoffs (or payoffs of Arrow-Debreu securities) in finance. For every coalition structure P from $\mathcal{P}(K)$ player i has a payoff function $U_i(P)\colon S(P) \to \mathbb{R}_+$, such that the set $U_i(P)$ is bounded, $U_i(P) < \infty$. Payoffs are considered as von Neumann-Morgenstern utilities. All payoffs of i for a game with no more than K deviators make family: $\mathcal{U}_i(K) = \{U_i(P)\colon P \in \mathcal{P}(K)\}$. Every coalition structure has it's own set of strategies and a corresponding set of payoffs. Thus every coalition structure is a non-cooperative game.

An increase in K increases the number of possible partitions and the set of strategies. Hence we obtain a nested family of payoff functions:

$$\mathcal{U}_i(K=1) \subset \ldots \subset \mathcal{U}_i(K) \subset \ldots \subset \mathcal{U}_i(K=N).$$

We can easily see that this construction of payoffs allows to obtain both intra and inter coalition (or group) externalities, as payoffs are defined directly over strategy profiles of all players.

Definition 2 (**a simultaneous coalition structure formation game**). *A non-cooperative game for coalition structure formation is*

$$\Gamma(K) = \Big\langle N, \Big\{K, \mathcal{P}(K), \mathcal{R}(K)\Big\}, \Big(S_i(K), \mathcal{U}_i(K)\Big)_{i \in N} \Big\rangle,$$

where $\Big\{K, \mathcal{P}(K), \mathcal{R}(K)\Big\}$ *- coalition structure formation mechanism (a social norm, a social institute),* $\Big(S_i(K), \mathcal{U}_i(K)\Big)_{i \in N}$ *- properties of players in* N*, (individual strategies and payoffs), such that:*

$$\times_{i \in N} S_i(K) \overset{\mathcal{R}(K)}{\to} \Big\{S(P) \colon P \in \mathcal{P}(K)\Big\} \to \Big\{(\mathcal{U}_i(K))_{i \in N}\Big\}.$$

Novelty of the paper is an introduction of coalition structure formation mechanism, which shapes the set of all strategies into non-cooperative partition-specific games. If we omit a middle of a formal definition of the game we obtain a non-cooperative game of Nash: $\times_{i \in N} S_i(K) \to (U_i)_{i \in N}$. Another novelty of the paper is an introduction of nested games.

Definition 3 (**family of games**). *A family of games is* ***nested*** *if :*

$$\mathcal{G} = \Gamma(K=1) \subset \ldots \subset \Gamma(K) \subset \ldots \subset \Gamma(K=N).$$

Nested games appear as a result of parametrization of a game by a maximum coalition size for the each game (or by a maximum number of deviators, what is equivalent). All games have consistent nesting of components, besides a set of players N.

Let $\Sigma_i(K)$ be a set of all mixed strategies of i, probability measures, $\Sigma_i(K) = \{\sigma_i(K)\colon \int_{S_i(K)} d\sigma_i(K) = 1\}$, with a general element $\sigma_i(K)$, where an integral is Lebegue integral.

Expected utility can be defined in terms of strategies the players choose or in terms of final partition-specific strategies. Expected utility of i:

$$EU_i^{\Gamma(K)}(\sigma_i(K), \sigma_{-i}(K)) = \int_{S(K)=\times_{i\in N} S_i(K)} U_i(s_i, s_{-i}) d\sigma_i(K) d\sigma_{-i}(K) \text{ or}$$

$$EU_i^{\Gamma(K)}(\sigma_i(K), \sigma_{-i}(K)) = \sum_{P\in\mathcal{P}(K)} \int_{S(P)} U_i(P)(s_i, s_{-i}) d\sigma_i(K) d\sigma_{-i}(K).$$

Expected utilities are constructed in the standard way.

Definition 4 (an equilibrium in a game $\Gamma(K)$). *A mixed strategies profile $\sigma^*(K) = (\sigma_i^*(K))_{i\in N}$ is an equilibrium strategy profile for a game $\Gamma(K)$ if for every $\sigma_i(K) \neq \sigma_i^*(K)$ the following inequality for every player i from N holds true:*

$$EU_i^{\Gamma(K)}\Big(\sigma_i^*(K), \sigma_{-i}^*(K)\Big) \geq EU_i^{\Gamma(K)}\Big(\sigma_i(K), \sigma_{-i}^*(K)\Big).$$

Equilibrium in the game $\Gamma(K)$ is defined in a standard way, it is just an expansion of Nash theorem. This result for non-cooperative games with coalition structure formation is different from the results of cooperative games, where an equilibrium may not exist, (for example, coalition form games with empty cores). Another outcome of the model is that there is no need to introduce additional properties of games, like axioms on a system of payoffs, super-additivity, weights. Equilibrium existence result can be generalized for the whole family of games.

Theorem 1. *The family of games $\mathcal{G} = \{\Gamma(K), K = 1, 2, \ldots, N\}$ has an equilibrium in mixed strategies, $\sigma^*(\mathcal{G}) = (\sigma^*(K = 1), \ldots, \sigma^*(K = N))$.*

This result is obvious. The theorem expands the classic Nash theorem. An equilibrium in the game can also be characterized by equilibrium partitions.

Definition 5 (equilibrium coalition structures or partitions). *A set of partitions $\{P^*\}(K)$, $\{P^*\}(K) \subset \mathcal{P}(K)$, of a game $\Gamma(K)$, is a set of equilibrium partitions, if it is induced by an equilibrium strategy profile $\sigma^*(K) = (\sigma_i^*(K))_{i\in N}$.*

In the example below we will see that there can be more than one equilibrium partition, and equilibrium partitions may change with an increase in the number of deviators. These issues are addressed more in the accompanying paper on applications of the game.

4 An example of a game of two players

The example serves to demonstrate a possible inefficiency of an equilibrium in a constructed game. The game can be considered as a generalization of the Prisoner's Dilemma.

There are 2 players. They can form two types of partitions. If $K = 1$ then there is only one final partition, $\mathcal{P}(K = 1) \equiv P_{separ} = \{\{1\}, \{2\}\}$. If $K = 2$ there are two final partitions, which make a family of partitions $\mathcal{P}(K = 2) = \{P_{separ}, P_{joint} = \{1, 2\}\}$. Partition structures $\mathcal{P}(K = 1)$ and $\mathcal{P}(K = 2)$ are nested.

Every player in every partition has two strategies: H(igh) and L(ow). Player i for a game with $K = 1$ has a strategy set $S_i(K = 1) = \{H_{i,P_{separ}}, L_{i,P_{separ}}\}$. Player i in the game with $K = 2$ has the strategy set $S_i(K = 2) = \{S_i(K = 1); H_{i,P_{joint}}, L_{i,P_{joint}}\}$. For player i strategy sets $S_i(K = 1)$ and $S_i(K = 2)$ are nested.

In Table 2 every cell in the table contains a payoff profile and a *final* coalition structure.

Set of strategies for the game with $K = 1$ is $S(K = 1) = \{H_{1,P_{separ}}, L_{1,P_{separ}}\} \times \{H_{2,P_{separ}}, L_{2,P_{separ}}\}$. Payoffs are in corresponding cells of Table 2, and are identical to the payoffs of the Prisoner's Dilemma.

Set of strategies for the game with $K = 2$ is $S(2) = \{S_1(K = 1); H_{1,P_{joint}}, L_{1,P_{joint}}\} \times \{S_2(K = 1); H_{2,P_{joint}}, L_{2,P_{joint}}\}$. Strategy sets of games with $K = 1$ and $K = 2$ are nested, i.e. $S(K = 1) \subset S(K = 2)$. Payoffs of the game for $K = 1$ are replicated, see Table 2.

A coalition structure formation mechanisms $\mathcal{R}(K = 1), \mathcal{R}(K = 2)$ follow the rule: the partition P_{joint} can be formed only from a unanimous agreement of both players. It is clear that $\mathcal{R}(K = 1) \subset \mathcal{R}(K = 2)$. This means that the grand coalition, or a partition P_{joint}. can be formed only over the strategy set $(H_{1,P_{joint}}, L_{1,P_{joint}}) \times (H_{2,P_{joint}}, L_{2,P_{joint}})$. Otherwise the partition P_{separ} is formed.

From Table 2 we can see that the whole strategy set of the game is partitioned into *coalition structure specific domains*. Every coalition structure (or a partition) is a non-cooperative game with it's own strategy set and payoff profiles. Final partition for a player may not coincide with an individual choice. A set of strategies for the partition P_{separ} is not a product of sets. Finally the games for $K = 1$ and $K = 2$ are nested.

Consider the game with a maximum coalition size $K = 2$ as described by the Table 2. If both players choose strategies only for the partition P_{separ}, then the game is the standard Prisoner's Dilemma game. However the same coalition structure may form if players disagree to form $P_{joint} = \{1, 2\}$. Thus for the *final* partition $P_{separ} = \{\{1\}, \{2\}\}$ there are three equilibria, and every equilibrium is inefficient. There are also three efficient and desirable outcomes in the same partition.

The partition $P_{joint} = \{1, 2\}$ can be formed only if both players choose it. Within this partition there is one an inefficient equilibrium and one efficient outcome.

Compare efficient outcomes for the partitions P_{separ} and P_{joint}. There are equal payoff profiles $(0;0)$ for the two partitions. But we can not address any of these efficient outcomes as cooperative, as they appear in different partitions: one in $P_{separ} = \{\{1\},\{2\}\}$ and another in $P_{joint} = \{1,2\}$. So the term "cooperation" need to be defined additionally, what is done in the next Section of the paper.

Using the same game we can demonstrate appearance of intra- and inter-coalitions externalities. If partition $P_{joint} = \{1,2\}$ is formed, then a payoff of a player depends on a strategy of another in the *same* coalition (presence of intra-coalition or intra-group externality). If partition $P_{separ} = \{\{1\},\{2\}\}$, is formed then payoff of a player depends on a strategy of another player in *the different* coalition (presence of inter-coalition or inter-group externality).

Multiplicity of equilibria makes both of these externalities co-exist in equilibria, but in different final coalition structures. Thus the game is able to present both intra and inter-coalition externalities, what is impossible in cooperative game theory.

Table 2: Payoff for the family of games with unanimous formation rules. Different partitions have payoff-equal efficient outcomes. What cooperation mean in these efficient outcomes?

	$L_{2,P_{separ}}$	$H_{2,P_{separ}}$	$L_{2,P_{joint}}$	$H_{2,P_{joint}}$
$L_{1,P_{separ}}$	$\underline{(0;0)}$: $\{\{1\},\{2\}\}$	(-5;3) : $\{\{1\},\{2\}\}$	$\underline{(0;0)}$: $\{\{1\},\{2\}\}$	(-5;3) : $\{\{1\},\{2\}\}$
$H_{1,P_{separ}}$	(3;-5): $\{\{1\},\{2\}\}$	**(-2;-2)**: $\{\{1\},\{2\}\}$	(3;-5) : $\{\{1\},\{2\}\}$	**(-2;-2)** : $\{\{1\},\{2\}\}$
$L_{1,P_{joint}}$	$\underline{(0;0)}$: $\{\{1\},\{2\}\}$	(-5;3) : $\{\{1\},\{2\}\}$	$\underline{(0;0)}$: $\{1,2\}$	$(-5;3)$: $\{1,2\}$
$H_{1,P_{joint}}$	(3;-5): $\{\{1\},\{2\}\}$	**(-2;-2)** : $\{\{1\},\{2\}\}$	$(3;-5)$: $\{1,2\}$	**(-2;-2)** : $\{1,2\}$

We can reinstall uniqueness of an equilibrium, what is done in Table 3. If both players are extroverts and prefer be together,[7] then every individual payoff increases by $\epsilon > 0$. This means that changing of the game, from $\Gamma(1)$ to $\Gamma(2)$ changes the equilibrium in terms of both strategies and the partition.

[7] what is equivalent to preferences over coalition structures

Table 3: Two players, $K = 1$ for P_{separ}, $K = 2$ for P_{separ}, P_{joint}

	$L_{2,P_{separ}}$	$H_{2,P_{separ}}$	$L_{2,P_{joint}}$	$H_{2,P_{joint}}$
$L_{1,P_{separ}}$	$\underline{(0;0)}$: $\{\{1\},\{2\}\}$	(-5;3) : $\{\{1\},\{2\}\}$	$\underline{(0;0)}$: $\{\{1\},\{2\}\}$	(-5;3) : $\{\{1\},\{2\}\}$
$H_{1,P_{separ}}$	(3;-5): $\{\{1\},\{2\}\}$	(-2;-2): $\{\{1\},\{2\}\}$	(3;-5) : $\{\{1\},\{2\}\}$	(-2;-2) : $\{\{1\},\{2\}\}$
$L_{1,P_{joint}}$	$\underline{(0;0)}$: $\{\{1\},\{2\}\}$	(-5;3) : $\{\{1\},\{2\}\}$	$\underline{(0+\epsilon; 0+\epsilon)}$ $\{1,2\}$	$(-5+\epsilon; 3+\epsilon)$ $\{1,2\}$
$H_{1,P_{joint}}$	(3;-5)$\{\{1\},\{2\}\}$	(-2;-2) : $\{\{1\},\{2\}\}$	$(3+\epsilon; -5+\epsilon)$ $\{1,2\}$	**(-2+ϵ;-2+ϵ)** $\{1,2\}$

If both players are introverts, $\epsilon < 0$, then the expansion of the game will not change initial equilibrium in terms of both strategies and the partition. However in both cases there is no changes in equilibrium payoff profiles. These issues are discussed in the accompanying paper.

5 Discussion

Insufficiency of cooperative game theory to study coalitions and coalition structures was earlier reported by many authors. Maskin (2011) wrote that "features of cooperative theory are problematic because most applications of game theory to economics involve settings in which externalities are important, Pareto inefficiency arises, and the grand coalition does not form". Myerson (p.370, 1991) noted that "we need some model of cooperative behavior that does not abandon the individual decision-theoretic foundations of game theory". Thus there is a demand for a specially designed non-cooperative game to study coalition structures formation along with an adequate equilibrium concept for this game.

There is a voluminous literature on the topic, a list of authors is far from complete: Aumann, Hart, Holt, Maschler, Maskin, Myerson, Peleg, Roth, Serrano, Shapley, Schmeidler, Weber, Winter, Wooders and many others.

A popular approach to use a "threat"as a basic concept for coalition formation analysis was suggested by Nash (1953). Consider a strategy profile from a subset of players. Let this profile be a threat to someone, beyond this subset. The threatening players may produce externalities for each other (and negative externalities not excluding!). How credible could be such threat? At the same time there may be some other player beyond the subset of players who may obtain a bonanza from this threat. But this beneficiant may not join the group due to expected intra-group negative externalities for members or from members of this group. Thus a concept of a threat can not serve as an elementary concept.

The justification of a chosen tool, a non-cooperative game, comes from Maskin (2011) and a remark of Serrano (2014), that for studying coalition formation ""quotedblleft it may be worth to use strategic-form games, as proposed in the Nash program".

The difference of the research agenda in this paper from the Nash program (Serrano 2004) is studying non-cooperative formation of coalition structures, but not only formation of one coalition. The best analogy for the difference is the difference between partial and general equilibrium analysis in economics. The former isolates a market ignoring cross-market interactions, the latter explicitly studies cross market interactions.

The constructed finite non-cooperative game allows to study what can be a cooperative behavior, when the individuals "rationally further their individual interests" (Olson, 1971).

Nash (1950, 1951) suggested to construct a non-cooperative game as a mapping of a set of strategies into a profile of payoffs, $\times_{i\in N} S_i \to (U_i)_{i\in N}$.

This paper has two contributions in comparison to his paper: construction of a non-cooperative game with an embedding coalition structure formation mechanism, and parametrization of all constructed games by a number of deviators: $\times_{i\in N} S_i(K) \overset{\mathcal{R}(K)}{\to} \{S(P) \colon P \in \mathcal{P}(K)\} \to \{(\mathcal{U}_i(K))_{i\in N}\}$, where $K \in \{1, N\}$. The game suggested by Nash becomes a partial case for these games.

Every game in a family has an equilibrium, may be in mixed strategies. This differs from results of cooperative game theory, where games may have no equilibrium, like games with empty cores, etc.

The introduced equilibrium concept differs from the strong Nash, coalition-proof and k equilibrium concepts. The differences are: an explicit allocation of payoffs and a combined presence of intra- and inter- coalition (or group) externalities (the list of differences is not complete). Differences from the core approach of Aumann (1960) are clear: a presence of externalities, no restrictions that only one group deviates, no restrictions on the direction of a deviation (inside or outside), and a construction of individual payoffs from a strategy profile of all players. The approach allows to study coalition structures, which differ from the grand coalition as in Shapley value. Finally the introduced concepts enables to offer a non-cooperative necessary stability criterion, ininfeasible for other equilibrium concepts, what is done in a sequential paper.

The suggested approach is different in a role for a central planner offered by Nash, who "argued that cooperative actions are the result of some process of bargaining" Myerson (p.370, 1991). The central planner offers a predefined coalition structure formation mechanism, that includes a maximum number of deviators, family of eligible partitions and a family of rules to construct these partitions from individual strategies of players.

The accompanying papers will demonstrate application of the suggested model for studying stochastic Bayesian games, cooperation, self-enforcement properties of an equilibrium (Aumann, 1990), non-cooperative criterion of partition stability, focal points and application of the same mechanism to study network games.

References

[1] Aumann, R., 1959. Acceptable points in general cooperative n-person games. Contributions to the theory of games. Annals of mathematic studies, no. 40, Vol. IV, Princeton University Press, Princeton, NJ. FINISH!

[2] Aumann R., 1990. Nash equilibria are not self-enforcing. Economic Decision Making: Games, Econometrics and Optimization, 201-206

[3] Aumann, R. J., Dreze J. H., 197. Cooperative games with coalition structures. International Journal of game theory 3 (4), 217-237

[4] Bernheim, B. D., Peleg, B., Whinston, M. D., 1987. Coalition-proof Nash equilibria i. concepts. Journal of Economic Theory 42, 1-12

[5] Hart, S., Kurz, M., 1983. Endogenous formation of coalitions. Econometrica: Journal of the Econometric Society, 1047-1064

[6] Holt, C., Roth, A., 2004. The Nash equilibrium: A perspective, Proceedings of the National Academy of Science 101(12), 3999-4002

[7] Myerson, R. B., 1977. Graphs and Cooperation in Games. Mathematics of Operations Research 2, 225-229.

[8] Maskin, E., 2011. Commentary: Nash equilibrium and mechanism design. Games and Economic Behavior 71(1), 9-11

[9] Nash, J. F., 1950. Equilibrium points in n-person games. Proceedings of the National Academy of Sciences, USA 36, 48-49.

[10] Nash, J.F., 1951. Non-cooperative games. Annals of Mathematics 54, 286-295

[11] Olson M., 2009, The logic of collective action. Cambridge: Harvard University Press.

[12] Serrano, R., 2004. Fifty years of the Nash program, 1953-2003