On the clonal approach in the mathematical theory of social choice

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Mathematical methods of decision analysis in economics, finance and politics, HSE
16 November 2016

1. Arrow's impossibility theorem
2. Shelah's extension
3. Complete classification of symmetric sets of r-choice function without the Arrow property
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Arrow's impossibility theorem

Arrow's impossibility theorem. Notation

- A – a non-empty (finite) set (of alternatives);
- n – a natural number (of voters), $n \geq 1$;
- (*individual*) *preferences* = (strict) linear order on A;
- $\mathrm{Ord}(A)$ – the set of all (strict) linear orders on A;
- *profile* = n-tuple of linear orders on A;
- (*universal*) *aggregation rule* = function $f : (\mathrm{Ord}(A))^n \to \mathrm{Ord}(A)$.

Arrow's impossibility theorem. Definitions

Definition

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An aggregation rule $f : (\mathrm{Ord}(A))^n \to \mathrm{Ord}(A)$ *satisfies*

U *(unanimity) iff*

$$(\forall a, b \in A)\,((\forall i < n)\, a \prec_i b) \;\to a\, f(\pi)\, b$$

for any profile $\pi = (\prec_0, \prec_1, \ldots, \prec_{n-1})$ *in* $(\mathrm{Ord}(A))^n$;

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for any profile $\pi = (\prec_0, \prec_1, \ldots, \prec_{n-1})$ *in* $(\mathrm{Ord}(A))^n$*;*

IIA *(independence of irrelevant alternatives) iff*

$$(\forall a, b \in A)\,((\forall i < n)\, a \prec_i b \leftrightarrow a \prec'_i b) \to (a\, f(\pi)\, b \leftrightarrow a\, f(\pi')\, b)$$

for all profiles $\pi = (\prec_0, \prec_1, \ldots, \prec_{n-1})$, $\pi' = (\prec'_0, \prec'_1, \ldots, \prec'_{n-1})$ *in* $(\mathrm{Ord}(A))^n$*.*

Arrow's impossibility theorem

An aggregation rule $f : (\mathrm{Ord}(A))^n \to \mathrm{Ord}(A)$ *satisfies*

D *(dictatorchip) iff* f *is a projection, i.e. there is* $i < n$ *such that for all* $\pi = (\prec_0, \prec_1, \ldots, \prec_{n-1}) \in (\mathrm{Ord}(A))^n$

$$f(\pi) = \prec_i .$$

Theorem (K. Arrow, 1950,1963)

For any natural number $n \geq 1$, *finite set* A *of cardinality* $|A| \geq 3$, *and aggregation rule* $f : (\mathrm{Ord}(A))^n \to \mathrm{Ord}(A)$ *if* f *satisfies* U *and* IIA *then* f *satisfies* **D**.

Arrow's impossibility theorem in terms of choice functions. Notation and definitions

- A – a (finite) set (of alternatives);
- $[A]^2 = \{B \subseteq A : |B| = 2\}$;
- 2-choice function on A – function $\mathfrak{c} : [A]^2 \to A$ satisfying

$$(\forall p \in [A]^2)\, \mathfrak{c}(p) \in p.$$

Definition

A 2-choice function $\mathfrak{c}$ on A is rational iff there is a (strict) linear order $\prec$ on A such that $\mathfrak{c}(p) = \max_{\prec} p$ for all $p \in [A]^2$, i.e.

$$\mathfrak{c}(\{a, b\}) = b \leftrightarrow a \prec b$$

for all different $a, b \in A$.

Arrow's impossibility theorem in terms of choice functions. Notation and definitions

- (*individual*) *preferences* = rational 2-choice function on A;
- $\mathfrak{R}_2(A)$ – the set of all rational 2-choice functions on A;
- n – a natural number (of voters);
- *profile* = n-tuple of rational 2-choice functions on A;
- (*universal*) *aggregation rule* = function $f : (\mathfrak{R}_2(A))^n \to \mathfrak{R}_2(A)$.

Note

The function Θ that assigns to each strict linear order $\prec$ the rational 2-choice function $\max_\prec$ is a bijection between $\mathrm{Ord}(A)$ and $\mathfrak{R}_2(A)$.

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U *(unanimity) iff*

$$(\forall p \in [A]^2)\,(\forall b \in p)\,((\forall i < n)\,\mathfrak{c}(p) = b) \to f(\pi)(p) = b$$

for any profile $\pi = (\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1})$ *in* $(\mathfrak{R}_2(A))^n$*;*

Arrow's impossibility theorem in terms of choice functions. Notation and definitions

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for any profile $\pi = (\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1})$ *in* $(\mathfrak{R}_2(A))^n$*;*

IIA *(independence of irrelevant alternatives) iff*

$$(\forall p \in [A]^2)((\forall i < n)\,\mathfrak{c}_i(p) = \mathfrak{c}_i'(p)) \to f(\pi)(p) = f(\pi')(p)$$

for all profiles $\pi = (\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1})$, $\pi' = (\mathfrak{c}_0', \mathfrak{c}_1', \ldots, \mathfrak{c}_{n-1}')$ *in* $(\mathfrak{R}_2(A))^n$*.*

Arrow's impossibility theorem in terms of choice functions. Notation and definitions

Note

A function $f : (\mathfrak{R}_2(A))^n \to \mathfrak{R}_2(A)$ *satisfies* IIA *iff for all* $p \in [A]^2$ *there is a function* $f_p : p^n \to p$ *such that*

$$f(\mathfrak{c}_0, \mathfrak{c}_1, \dots, \mathfrak{c}_{n-1})(p) = f_p(\mathfrak{c}_0(p), \mathfrak{c}_1(p), \dots, \mathfrak{c}_{n-1}(p))$$

for all $p \in [A]^2$ *and* $\mathfrak{c}_0, \mathfrak{c}_1, \dots, \mathfrak{c}_{n-1} \in \mathfrak{R}_2(A)$.
A function $f : (\mathfrak{R}_2(A))^n \to \mathfrak{R}_2(A)$ f *satisfies* IIA *and* U *iff there is a conservative function* $\widehat{f} : A^n \to A$ *such that*

$$f(\mathfrak{c}_0, \mathfrak{c}_1, \dots, \mathfrak{c}_{n-1})(p) = \widehat{f}(\mathfrak{c}_0(p), \mathfrak{c}_1(p), \dots, \mathfrak{c}_{n-1}(p)).$$

for all $p \in [A]^2$ *and* $\mathfrak{c}_0, \mathfrak{c}_1, \dots, \mathfrak{c}_{n-1} \in \mathfrak{R}_2(A)$.

Conservative functions

Definition

A function $g : A^n \to A$ is conservative iff

$$(\forall x_0, x_1, \ldots, x_{n-1} \in A) \bigvee_{i<n} (g(x_0, x_1, \ldots, x_{n-1}) = x_i).$$

Arrow's impossibility theorem in terms of choice functions

An aggregation rule $f : (\mathrm{Ord}(A))^n \to \mathrm{Ord}(A)$ *satisfies*

D *iff* f *is a projection, i.e. there is* $i < n$ *such that for all* $\pi = (\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1}) \in (\mathfrak{R}_2(A))^n$

$$f(\pi) = \mathfrak{c}_i.$$

Theorem

For any natural number $n \geq 1$*, finite set* A *of cardinality* $|A| \geq 3$*, and aggregation rule* $f : (\mathfrak{R}_2(A))^n \to \mathfrak{R}_2(A)$ *if* f *satisfies* U *and* IIA *then* f *satisfies* $\mathbf{D}$*.*

Shelah's extension

Shelah's extension. Notation and definitions

- A – a non-empty (finite) set (of alternatives);
- r – a natural number (technical parameter), $r \geq 1$;
- $[A]^r = \{B \subseteq A : |B| = r\}$;
- r-choice function on A – function $\mathfrak{c} : [A]^r \to A$ satisfying

$$(\forall p \in [A]^r)\, \mathfrak{c}(p) \in p.$$

- $\mathfrak{C}_r(A)$ – the set of all r-choice function on A;

Definition

A set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ is symmetric if for any function $\mathfrak{c} \in \mathfrak{D}$ and permutation $\sigma \in S_A$ the function $\mathfrak{c}_\sigma$ defined by

$$(\forall p \in [A]^r)\, \mathfrak{c}_\sigma(p) = \sigma^{-1}\mathfrak{c}(\sigma p),$$

belongs to $\mathfrak{D}$.

Symmetric sets of r-choice functions

Exemples

- *The set* $\mathfrak{R}_2(A)$.
- *The set of all function* $\mathfrak{c} \in \mathfrak{C}_r(A)$ *such that* $\mathfrak{c}(p)$ *is the median element in* p *according to some ordering (*r *is odd).*
- *The set* $\{\mathfrak{c} \in \mathfrak{C}_2(A) \colon (\exists x \in A)(\forall y \in A \setminus \{x\})\, \mathfrak{c}(\{x, y\}) = x\}$.
- *Let* $\prec$ *be a strict partial order on* A *and* $\mathfrak{C}_r^{\prec}(A)$ *a set of all functions* $\mathfrak{c} \in \mathfrak{C}_r(A)$ *such that* $\mathfrak{c}(p)$ *is some non-dominated element of* p*, i.e.*

  $$(\forall x \in p)\, \mathfrak{c}(p) \not\prec x.$$

  Let W *be a set of strict partial order on* A *closed under isomorphisms. The set* $\bigcup\limits_{\prec \in W} C_r^{\prec}(A)$ *is symmetric.*

Shelah's extension. Notation and definitions

- (*individual*) *preferences* = r-choice function on A;
- n – a natural number (of voters), $n \geq 1$;
- *profile* = n-tuple of r-choice functions on A;
- *aggregation rule* = function $f : (\mathfrak{C}_r(A))^n \to \mathfrak{C}_r(A)$;
- $\mathcal{V}(A, r)$ – the set of all aggregation rules (of all arity $n \geq 1$).

Shelah's extension. Notation and definitions

Definition

An n-ary aggregation rule $f \in \mathcal{V}(A, r)$ is normal iff

(i) $f(\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1})(q) \in \{\mathfrak{c}_0(q), \mathfrak{c}_1(q), \ldots, \mathfrak{c}_{n-1}(q)\}$ *for all* $\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1} \in \mathfrak{C}_r(A)$ *and* $q \in [A]^r$*;*

Shelah's extension. Notation and definitions

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(i) $f(\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1})(q) \in \{\mathfrak{c}_0(q), \mathfrak{c}_1(q), \ldots, \mathfrak{c}_{n-1}(q)\}$ *for all* $\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1} \in \mathfrak{C}_r(A)$ *and* $q \in [A]^r$;

(ii) $(\mathfrak{c}_0(q), \mathfrak{c}_1(q), \ldots, \mathfrak{c}_{n-1}(q)) = (\mathfrak{c}'_0(q), \mathfrak{c}'_1(q), \ldots, \mathfrak{c}'_{n-1}(q)) \rightarrow f(\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1})(q) = f(\mathfrak{c}'_0, \mathfrak{c}'_1, \ldots, \mathfrak{c}'_{n-1})(q)$ *for all* $\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1}, \mathfrak{c}'_0, \mathfrak{c}'_1, \ldots, \mathfrak{c}'_{n-1} \in \mathfrak{C}_r(A)$ *and* $q \in [A]^r$.

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- $\mathcal{N}(A, r)$ – the set of all normal aggregation rules in $\mathcal{V}(A, r)$.

Shelah's extension. Notation and definitions

Definition

An n-ary aggregation rule $f \in \mathcal{V}(A, r)$ is simple iff

$$(\mathfrak{c}_0(p), \mathfrak{c}_1(p), \dots, \mathfrak{c}_{n-1}(p)) = (\mathfrak{c}_0(q), \mathfrak{c}_1(q), \dots, \mathfrak{c}_{n-1}(q)) \rightarrow$$
$$f(\mathfrak{c}_0, \mathfrak{c}_1, \dots, \mathfrak{c}_{n-1})(p) = f(\mathfrak{c}_0, \mathfrak{c}_1, \dots, \mathfrak{c}_{n-1})(q)$$

for all $\mathfrak{c}_0, \mathfrak{c}_1, \dots, \mathfrak{c}_{n-1} \in \mathfrak{C}_r(A)$ and $p, q \in [A]^r$.

Shelah's extension. Notation and definitions

Definition

An n-ary aggregation rule $f \in \mathcal{V}(A,r)$ is simple iff

$$(\mathfrak{c}_0(p), \mathfrak{c}_1(p), \ldots, \mathfrak{c}_{n-1}(p)) = (\mathfrak{c}_0(q), \mathfrak{c}_1(q), \ldots, \mathfrak{c}_{n-1}(q)) \to f(\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1})(p) = f(\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1})(q)$$

for all $\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1} \in \mathfrak{C}_r(A)$ and $p, q \in [A]^r$.

- $\mathcal{S}(A,r)$ – the set of all simple aggregation rules in $\mathcal{V}(A,r)$.

Shelah's extension. Notation and definitions

Definition

An n-ary aggregation rule $f \in \mathcal{V}(A, r)$ is a dictatorship iff it is a projection, i.e. iff there is $i < n$ such that $f(\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1}) = \mathfrak{c}_i$ for all $\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1} \in \mathfrak{C}_r(A)$

Shelah's extension. Notation and definitions

Definition

An n-ary aggregation rule $f \in \mathcal{V}(A, r)$ is a dictatorship *iff it is a projection, i.e. iff there is $i < n$ such that $f(\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1}) = \mathfrak{c}_i$ for all $\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1} \in \mathfrak{C}_r(A)$*

- $\mathcal{E}(A, r)$ – the set of all dictatorships in $\mathcal{V}(A, r)$.

Shelah's extension. Notation and definitions

Note

An n-ary function $f \in \mathcal{V}(A,r)$ is normal iff for all $p \in [A]^2$ there is a conservative function $f_p : p^n \to p$ such that

$$f(\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1})(p) = f_p(\mathfrak{c}_0(p), \mathfrak{c}_1(p), \ldots, \mathfrak{c}_{n-1}(p))$$

for all $p \in [A]^2$ and $\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1} \in \mathfrak{C}_r(A)$.
An n-ary function $f : (\mathfrak{R}_2(A))^n \to \mathfrak{R}_2(A)$ f is normal and simple iff there is a conservative function $\widehat{f} : A^n \to A$ such that

$$f(\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1})(p) = \widehat{f}(\mathfrak{c}_0(p), \mathfrak{c}_1(p), \ldots, \mathfrak{c}_{n-1}(p)).$$

for all $p \in [A]^2$ and $\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1} \in \mathfrak{C}_r(A)$.

Shelah's extension. Notation and definitions

Definition

An n-ary aggregation rule $f \in \mathcal{V}(A, r)$ preserves a set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ (or f is a polymorphism of $\mathfrak{D}$) and $\mathfrak{D}$ is preserved under f iff

$$f(\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n) \in \mathfrak{D} \text{ for all } \mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n \in \mathfrak{D}.$$

The set of all $f \in \mathcal{V}(A, r)$ that preserves $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ is denoted by $\operatorname{pol} \mathfrak{D}$.

Shelah's extension. Notation and definitions

Definition

A set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$

- *has the Arrow property iff*

$$\operatorname{pol}\mathfrak{D} \cap \mathcal{N}(A, r) = \mathcal{E}(A, r).$$

- *has the simple Arrow property iff*

$$\operatorname{pol}\mathfrak{D} \cap \mathcal{N}(A, r) \cap \mathcal{S}(A, r) = \mathcal{E}(A, r).$$

Shelah's theorem on the Arrow property

Theorem (S. Shelah, 2005)

Let A be a finite set. Then there are natural numbers r_1, r_2 (e.g. $r_1 = r_2 = 7$) such that for any natural number r, $r_1 \leq r \leq |A| - r_2$, any non-empty proper symmetric subset $\mathfrak{D}$ of the set $\mathfrak{C}_r(A)$ has the Arrow property.

Complete classification of symmetric sets of r-choice function without the Arrow property

Exceptional cases: $\mathfrak{C}_3^K(A)$

Let $|A| = 4$ and let K be the *Klein four-group* of permutations of A. For any sets $p, q \in [A]^3$ there is only one permutation $\sigma_{p,q} \in K$ for which

$$q = \sigma_{p,q}(p).$$

- $\mathfrak{C}_3^K(A)$ is the set of all functions $\mathfrak{c} \in \mathfrak{C}_3(A)$ such that

$$\mathfrak{c}(q) = \sigma_{p,q}\mathfrak{c}(p) \text{ for all } p, q \in [A]^3.$$

Exceptional cases: $\mathfrak{C}_3^K(A)$

The set $\mathfrak{C}_3^K(A)$ is symmetric and it contains exactly three elements $\mathfrak{c}_0, \mathfrak{c}_1, \mathfrak{c}_2$ (we denote $A = \{a, b, c, d\}$):

q	$\mathfrak{c}_0(q)$	$\mathfrak{c}_1(q)$	$\mathfrak{c}_2(q)$
$\{a, b, c\}$	a	b	c
$\{a, b, d\}$	b	a	d
$\{a, c, d\}$	c	d	a
$\{b, c, d\}$	d	c	b

Exceptional cases: $\mathfrak{C}_3^K(A)$

Let the conservative function $w\colon A^2 \to A$ be defined by

w	a	b	c	d
a	a	a	c	d
b	b	b	c	d
c	a	b	c	c
d	a	b	d	d

Let the 2-ary simple normal aggregation rule $f \in \mathcal{V}(A,3)$ be defined by

$$f_q(x,y) = w(x,y)$$

for all $q \in [A]^3$ and $x, y \in q$.

Exceptional cases: $\mathfrak{C}_3^K(A)$

We have

f	$\mathfrak{c}_0$	$\mathfrak{c}_1$	$\mathfrak{c}_2$
$\mathfrak{c}_0$	$\mathfrak{c}_0$	$\mathfrak{c}_0$	$\mathfrak{c}_2$
$\mathfrak{c}_1$	$\mathfrak{c}_1$	$\mathfrak{c}_1$	$\mathfrak{c}_2$
$\mathfrak{c}_2$	$\mathfrak{c}_0$	$\mathfrak{c}_1$	$\mathfrak{c}_2$

Proposition

The set $\mathfrak{C}_3^K(A)$ is preserved by the normal and simple aggregation rule f. So, $\mathfrak{C}_3^K(A)$ is non-empty proper symmetric subset of the set $\mathfrak{C}_3(A)$ without the simple Arrow property.

Exceptional cases: $\mathfrak{C}_2^i(A)$

Let $r = 2$ and $|A| \geq 2$. For any $a \in A$, $i \in \{0, 1\}$ and $\mathfrak{c} \in \mathfrak{C}_2(A)$

$$Z_a^{\mathfrak{c}} = \{b \in A \setminus \{a\} \colon \mathfrak{c}(\{a, b\}) = a\},$$
$$W_i^{\mathfrak{c}} = \{a \in A \colon |Z_a^{\mathfrak{c}}| = i \pmod 2\}.$$

- $\mathfrak{C}_2^i(A) = \{\mathfrak{c} \in \mathfrak{C}_2(A) \colon W_{(1-i)}^{\mathfrak{c}} = \varnothing\}.$

Exemple

$A = \{a, b, c\}$, $\mathfrak{C}_2^1(A) = \{\mathfrak{c}_0, \mathfrak{c}_1\}$.

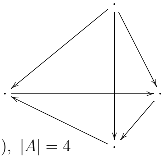
q	$\mathfrak{c}_0(q)$	$\mathfrak{c}_1(q)$
$\{a, b\}$	b	a
$\{b, c\}$	c	b
$\{a, c\}$	a	c

Exceptional cases: $\mathfrak{C}_2^i(A)$

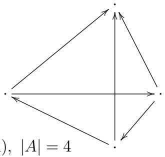
Another definition. Any function $\mathfrak{c} \in \mathfrak{C}_2(A)$ may be represented by the *tournament* $\Gamma_{\mathfrak{c}} = (A, E)$ where

$$E = \{(a, b) \in A^2 \colon a \neq b \wedge \mathfrak{c}(\{a, b\}) = b\}.$$

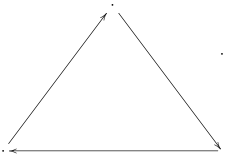
- The sets $\mathfrak{C}_2^0(A)$ and $\mathfrak{C}_2^1(A)$ are the sets of all functions $\mathfrak{c} \in \mathfrak{C}_2(A)$ such that the *indegree* of any node of the tournament $\Gamma_{\mathfrak{c}}$ is even (respectively, odd).



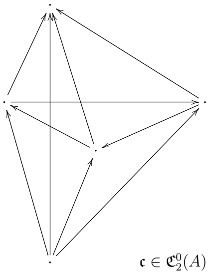
$\mathfrak{c} \in \mathfrak{C}_2^0(A),\ |A| = 4$



$\mathfrak{c} \in \mathfrak{C}_2^1(A),\ |A| = 4$



$\mathfrak{c} \in \mathfrak{C}_2^1(A),\ |A| = 3$



$\mathfrak{c} \in \mathfrak{C}_2^0(A),\ |A| = 5$

Exceptional cases: $\mathfrak{C}_2^i(A)$

For any non-empty set A the 3-ary normal simple aggregation rule $\ell^A \in \mathcal{V}(A, 2)$ is defined by

$$\ell_q^A(x, x, y) = \ell_q^A(x, y, x) = \ell_q^A(y, x, x) = y$$

for all $q \in [A]^2$ and $x, y \in q$.

Proposition

Each of the sets $\mathfrak{C}_2^0(A)$, $\mathfrak{C}_2^1(A)$, and $\mathfrak{C}_2^0(A) \cup \mathfrak{C}_2^1(A)$ is preserved by the normal and simple aggregation rule ℓ^A. So, it does not has the simple Arrow property. Besides, each of these sets is symmetric, and

1. $\mathfrak{C}_2^0(A) \neq \varnothing \leftrightarrow |A| = 0, 1 \pmod 4$;
2. $\mathfrak{C}_2^1(A) \neq \varnothing \leftrightarrow |A| = 0, 3 \pmod 4$;
3. $\mathfrak{C}_2^0(A) \cup \mathfrak{C}_2^1(A) \neq \mathfrak{C}_2(A)$.

Complete classification of symmetric sets of r-choice function without the Arrow property

Theorem (N. Polyakov, 2014)

Let A be a finite set, r be a natural number, and $\mathfrak{D}$ be a non-empty proper symmetric subset of the set $\mathfrak{C}_r(A)$. Then the set $\mathfrak{D}$ does not has the Arrow property if and only if one of the following conditions holds:

1. $r = 2$, $|A|$ *equals* 0 *or* $1 \pmod 4$, *and* $\mathfrak{D} = \mathfrak{C}_2^0(A)$,
2. $r = 2$, $|A|$ *equals* 0 *or* $3 \pmod 4$, *and* $\mathfrak{D} = \mathfrak{C}_2^1(A)$,
3. $r = 2$, $|A| = 0 \pmod 4$, *and* $\mathfrak{D} = \mathfrak{C}_2^0(A) \cup \mathfrak{C}_2^1(A)$,
4. $r = 3$, $|A| = 4$, *and* $\mathfrak{D} = \mathfrak{C}_3^K(A)$.

Arrow property for classes of decision rules

Arrow property for classes of decision rules. Notation and definitions

- Q – a non-empty (finite) set (of conditions);
- A – a non-empty (finite) set (of solutions);
- $\mathfrak{D} \subseteq {}^{Q}A$ – a set (of decision rules).

Definition

A set $\mathfrak{D} \subseteq {}^{Q}A$ is (weakly) symmetric iff for any permutation σ of A there is a permutation σ^* of Q such that for all function $\mathfrak{c} \in \mathfrak{D}$ the functions $\mathfrak{c}_\sigma$ and $\widetilde{\mathfrak{c}}_\sigma$ defined by

$$\mathfrak{c}_\sigma(q) = \sigma^{-1}\mathfrak{c}(\sigma^* q) \text{ and } \widetilde{\mathfrak{c}}_\sigma(q) = \sigma\mathfrak{c}\left((\sigma^*)^{-1} q\right)$$

belong to $\mathfrak{D}$.

Arrow property for classes of decision rules. Notation and definitions

Exemples

- $Q = [A]^r$, $\mathfrak{D} = \mathfrak{C}_r(A)$;
- $Q = \mathcal{P}(A) \setminus \{\varnothing\}$, $\mathfrak{D}$ *is a set of all (total) choice functions;*
- Q *is a set of subsets of* A *enriched with some additional structure, for example*
  - Q *is a set of all linear orders on* A,
  - Q *is a set of non-empty multisets such that the underlying set of elements is a subset of* A,

  and $\mathfrak{D}$ *is a set of choice function;*
- Q *is a set of all linear orders on some set* B, $A = \mathcal{P}(B)$, $\mathfrak{D} = {}^{Q}A$.

Arrow property for classes of decision rules. Notation and definitions

- n – a natural number (of voters), $n \geq 1$;
- *profile* $=$ n-tuple of decision rules in $\mathfrak{D}$;
- *(simple) aggregation rule* $=$ conservative function $f : A^n \to A$;
- $\mathcal{V}(A)$ – the set of all simple aggregation rules (of all arity $n \geq 1$).

Arrow property for classes of decision rules. Notation and definitions

Definition

A function $f : A^n \to A$ *preserves a set* $\mathfrak{D} \subseteq {}^Q A$ *and* $\mathfrak{D}$ *is preserved under* f *iff for all* $\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1} \in \mathfrak{D}$ *the set* $\mathfrak{D}$ *contains the function* $f(\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1})$ *defined by*

$$f(\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1})(q) = f(\mathfrak{c}_0(q), \mathfrak{c}_1(q), \ldots, \mathfrak{c}_{n-1}(q))$$

for all $q \in Q$.

Arrow property for classes of decision rules. Notation and definitions

- $\operatorname{pol}\mathfrak{D}$ – the set of all functions $f : A^n \to A$ (of any arity n) that preserve a set $\mathfrak{D} \subseteq {}^{Q}A$;
- $\operatorname{inv}_Q f$ – the set of all sets $\mathfrak{D} \subseteq {}^{Q}A$ that is preserved under a function $f : A^n \to A$;
- for all sets $\mathcal{F} \subseteq \bigcup\limits_{n<\omega} {}^{A^n}A$ and $\mathbb{D} \subseteq \mathcal{P}({}^{Q}A)$

$$\operatorname{inv}_Q \mathcal{F} = \bigcap_{f\in\mathcal{F}} \operatorname{inv}_Q f \text{ and } \operatorname{pol}\mathbb{D} = \bigcap_{\mathfrak{D}\in\mathbb{D}} \operatorname{pol}\mathfrak{D}.$$

Arrow property for classes of decision rules. Notation and definitions

Theorem

The couple $(\mathrm{inv}_Q, \mathrm{pol})$ *is a Galois correspondence between the Boolean lattices* $\mathcal{P}(\bigcup\limits_{n<\omega} {}^{A^n}A)$ *and* $\mathcal{P}(\mathcal{P}({}^{Q}A))$. *Galois-closed sets* $\mathcal{F} \subseteq \bigcup\limits_{n<\omega} {}^{A^n}A$ *are closed under composition and contain all projections, i.e. is clones. If a set* $\mathfrak{D} \subseteq {}^{Q}A$ *is symmetric, then the clone* $\mathrm{pol}\,\mathfrak{D}$ *is symmetric, i.e. for all permutation* σ *of* A *and* n*-ary function* $f \in \mathrm{pol}\,\mathfrak{D}$ *the clone* $\mathrm{pol}\,\mathfrak{D}$ *contains a function* $f_\sigma : A^n \to A$, *defined by*

$$f_\sigma(\mathbf{a}) = \sigma^{-1} f(\sigma\mathbf{a})$$

for all $\mathbf{a} \in A^n$.

Arrow property for classes of decision rules. Notation and definitions

Definition

The set $\mathfrak{D} \subseteq {}^{Q}A$ has a (simple) Arrow property iff for any natural number n and any n-ary function $f \in \mathrm{pol}\,\mathfrak{D} \cap \mathcal{V}(A)$ there is a number i ($i < n$) for which

$$(\forall \mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1} \in \mathfrak{D})\, f(\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1}) = \mathfrak{c}_i,$$

i.e.

$$f(\mathbf{a}) = a_i,$$

for all n-tuples $\mathbf{a} \in \{(\mathfrak{c}_0(q), \mathfrak{c}_1(q), \ldots, \mathfrak{c}_{n-1}(q)) : q \in Q, \mathfrak{c}_i \in \mathfrak{D}\}$.

Are there *many* symmetric sets of decision rules without Arrow properties?

Exemple

Let A be a finite set, r be a natural number, $2 \leq r \leq |A| - 1$, and $\mathfrak{D}$ be a symmetric subset of $\mathfrak{C}_r(A)$. Let

- $Q = [A]^r \cup [A]^{r+1}$;
- $\mathfrak{C}$ *be the set of all choice functions* $\mathfrak{c}$ *on* Q *such that* $\mathfrak{c} \restriction [A]^r \in \mathfrak{D}$.

Are there *many* symmetric sets of decision rules without Arrow properties?

Exemple

Then

- $\mathfrak{C}$ *is symmetric,* $\mathfrak{C} \restriction [A]^r = \mathfrak{D}$;
- $\mathfrak{C}$ *is preserved under the conservative function* $f : A^{r+1} \to A$ *defined by*

$$f(\mathbf{x}) = \begin{cases} x_1, & \textit{if } |\operatorname{ran}\mathbf{x}| = r+1; \\ x_0, & \textit{if } |\operatorname{ran}\mathbf{x}| \leq r. \end{cases}$$

for all $\mathbf{x} = (x_0, x_1, \ldots, x_r) \in A^{r+1}$.

Are there *many* symmetric sets of decision rules without Arrow properties?

Exemple

q	$\mathfrak{c}_0(q)$	$\mathfrak{c}_1(q)$	$\dots$	$\mathfrak{c}_r(q)$	$f(\mathfrak{c}_0, \mathfrak{c}_1, \dots, \mathfrak{c}_r)(q)$
$\{a_0, a_1, \dots, a_r\}$	a_0	a_1	$\dots$	a_r	a_1
$\{a_0, a_1, \dots, a_{r-1}\}$	a_0	a_1	$\dots$	a_1	a_0

$\mathfrak{C}$ *does not have the Arrow property.*

Are there *many* symmetric sets of decision rules without Arrow properties?

Theorem

Let A and Q be a finite sets, and $\mathfrak{D}$ be a symmetric subset of ${}^{Q}A$. Then there are finite sets $B \supseteq A$ and $P \supseteq Q$ and a symmetric set $\mathfrak{C} \subseteq {}^{P}B$ such that

- *$\mathfrak{C}$ does not have the Arrow property;*
- ${}^{Q}A \cap \mathfrak{C} \restriction Q = \mathfrak{D}$.

Arrow property for classes of decision rules. Notation and definitions

- $\mathbb{B}_0$ is the set of all sets $\mathfrak{B} \subseteq {}^Q A$ of the form
$$\{\mathfrak{c} \in {}^Q A : \mathfrak{c}(q) \in B\}$$
where $q \in Q$ and $B \subseteq A$;
- $\mathbb{B}_1$ is the set of all sets $\mathfrak{B} \subseteq {}^Q A$ of the form
$$\{\mathfrak{c} \in {}^Q A : \mathfrak{c}(p) = a \vee \mathfrak{c}(q) = b\},$$
where $p, q \in Q$ and $a, b \in A$;

Arrow property for classes of decision rules. Notation and definitions

- $\mathbb{B}_2(R)$ is the set of all sets $\mathfrak{B} \subseteq {}^QA$ of the form

$$\{\mathfrak{c} \in {}^QA : \mathfrak{c}(q) = \sigma\mathfrak{c}(p)\},$$

  where R is a binary relation on $A^{<\omega}$, $p, q \in Q$, $\sigma \in S_A$ and $(\mathbf{b}, \sigma\mathbf{b}) \in R$ for all $\mathbf{b} \in A^{<\omega}$;
- $\mathbb{B}_3(\Pi)$ is the set of all sets $\mathfrak{B} \subseteq {}^QA$ of the form

$$\{\mathfrak{c} \in {}^QA : \mathfrak{c} \restriction P \in \mathrm{inv}_P(\Pi_B)\},$$

  where Π is a Post's class closed under duality, $B \in [A]^2$, Π_B is a clone on B naturally isomorphic to Π, $P \subseteq Q$.

Arrow property for classes of decision rules. Notation and definitions

Note

There are only six Post's classes $\Pi \subseteq T_{01}$ *closed under duality:* O_1, D_1, D_2, L_4, A_4, T_{01}.

Definition

A binary relation R *on* $A^{<\omega}$ *is stable iff*

1. $\mathbf{a}\, R\, \mathbf{b} \to \mathbf{a} = \sigma\mathbf{b}$ *for some permutation* σ *of* A*;*
2. $\mathbf{a}\, R\, \mathbf{b} \to \sigma\mathbf{a}\tau\, R\, \sigma\mathbf{b}\tau$ *for any permutation* σ *of* A*, natural number* k *and function* $\tau\colon \{0, 1, \dots, k-1\} \to \operatorname{dom} \mathbf{a}$*.*

Arrow property for classes of decision rules. Notation and definitions

Let $\mathfrak{D} \subseteq {}^{Q}A$. For any natural number r

- $r(\mathfrak{D}) = \max_{q \in Q} |\{\mathfrak{c}(q) : \mathfrak{c} \in \mathfrak{D}\}|$
- $Q_{\mathfrak{D},r} = \{q \in Q : |\{\mathfrak{c}(q) : \mathfrak{c} \in \mathfrak{D}\}| \leq r\}$;
- $\mathfrak{D}_r = \mathfrak{D} \restriction Q_{\mathfrak{D},r}$
- $\mathfrak{D}_r^+ = \{\mathfrak{c} \in {}^{Q}A : \mathfrak{c} \restriction Q_{\mathfrak{D},r} \in \mathfrak{D}_r\}$;

Arrow property for classes of decision rules

Theorem

Let A and Q be non-empty finite sets. Let $\mathfrak{D} \subseteq {}^{Q}A$ be a symmetric set without Arrow property. Then there are a stable binary relation R on $A^{<\omega}$, a Post's class $\Pi \in \{O_1, D_1, D_2, L_4, A_4, T_{01}\}$ and a set $\mathbb{B} \subseteq \mathbb{B}_0 \cup \mathbb{B}_1 \cup \mathbb{B}_2(R) \cup \mathbb{B}_3(\Pi)$ such that one of two following conditions holds

1. $\mathfrak{D} = \bigcap \mathbb{B}$;
2. *there is a natural number $r < r(\mathfrak{D})$ such that $\mathfrak{D} = \mathfrak{D}_r^+ \cap \bigcap \mathbb{B}$, and any n-ary function $f \in \mathrm{pol}\, \mathfrak{D} \cap \mathcal{V}(A)$ coincides with a projection on the set $A^n_{\leq r} = \{\mathbf{a} \in A^n : |\mathrm{ran}\, \mathbf{a}| \leq r\}$ (hence $\mathfrak{D}_r$ has the Arrow property).*

Some positive results

Theorem

Let A be a non-empty finite set, $|A| \geq 3$, and f be an arbitrary non-dictatorship function in the clone $\mathcal{D}$ generated by a (majority) function ∂ satisfying

$$\partial(x, x, y) = \partial(x, y, x) = \partial(y, x, x) = x.$$

Then a set $\mathfrak{D} \subseteq \mathfrak{C}_2(A)$ belong of $\mathrm{inv}_{[A]^2} f$ if and only if the set $\mathfrak{D}$ is an intersection of a family of sets of the form

$$\{\mathfrak{c} \in \mathfrak{C}_2(A) \colon \mathfrak{c}(\{a, b\}) = a \to \mathfrak{c}(\{c, d\}) = d\}.$$

where $a, b, c, d \in A$.

Thank you!