A DIRECT FAST FFT-BASED IMPLEMENTATION FOR HIGH ORDER FINITE ELEMENT METHOD ON RECTANGULAR PARALLELEPIPEDS FOR PDE

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Abstract

We present a new direct logarithmically optimal in theory and fast in practice algorithm to implement the high order finite element method on multi-dimensional rectangular parallelepipeds for solving PDEs of the Poisson kind. The key points are the fast direct and inverse FFT-based algorithms for decomposition in eigenvectors of the 1D eigenvalue problems for the high order FEM. The algorithm can further be used for numerous applications, in particular, to implement the high order finite element methods for various time-dependent PDEs.

Key words. Fast direct algorithm, high order finite element method, FFT, Poisson equation.

AMS subject classifications. 65F05, 65F15, 65M60, 65T99.

arXiv:1609.07758v1 [math.NA] 25 Sep 2016

1 Introduction

We present new direct fast algorithm to implement nth order ($n \geqslant 2$) finite element method (FEM) on rectangular parallelepipeds [3] for solving N-dimensional PDEs, $N \geqslant 2$, like the Poisson one with the Dirichlet boundary condition. The algorithm generalizes the well-known one in the case of the bilinear elements ($n = 1$) or standard finite-difference schemes [1, 7, 8] and utilizes the discrete fast Fourier transforms (FFTs) [2]. The key points are the fast direct and inverse algorithms for decomposition in eigenvectors of the 1D eigenvalue problems for the high order FEM; this solves the known problem, see [1, p. 271]. The algorithm is logarithmically optimal with respect to the number of elements. It also demonstrates rather mild growth in n starting from the known case $n = 1$ and is fast in practice, for example, the 2D FEM system for 2^{20} elements of the 9th order containing almost $85 \cdot 10^6$ unknowns is solved in less than 2 min on an ordinary laptop, see Fig. 1 below. The algorithm can further serve for a variety of applications including general 2nd order elliptic equations (as a preconditioner), for the N-dimensional heat, wave or time-dependent Schrödinger PDEs. It can be applied for some non-rectangular domains, in particular, by involving meshes topologically equivalent to rectangular ones [6]. Other standard boundary conditions can be covered as well [10]; moreover, the structure of the algorithm is valuable for wave problems with non-local boundary conditions, see [1, 4, 5, 9], whence our own interest arose. The algorithm is also highly parallelizable.

2 Algorithms

1. We first need to consider in detail the FEM for the simplest 1D eigenvalue ODE problem

$$-u''(x) = \lambda u(x) \ \text{ on } \ [0, X], \quad u(0) = u(X) = 0, \quad u(x) \not\equiv 0. \tag{1}$$

We introduce the uniform mesh with the nodes $x_j = jh$, $j = \overline{0, K}$ (i.e., $0 \leqslant j \leqslant K$) and the step $h = X/K$. Let $H_h^{(n)}[0, X]$ be the space of the piecewise-polynomial functions $\varphi \in C[0, X]$ such that $\varphi(x) \in \mathcal{P}_n$ for $x \in [x_{j-1}, x_j]$, $j = \overline{1, K}$, with $\varphi(0) = \varphi(X) = 0$; here $\mathcal{P}_n$ is the space of polynomials having at most nth degree, $n \geqslant 2$.

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Let $S_K^{(n)}$ be the space of vector functions w such that $w_j \in \mathbb{R}$ for $j = \overline{0,K}$ with $w_0 = w_K = 0$ and $w_{j-1/2} \in \mathbb{R}^{n-1}$, $j = \overline{1,K}$. Clearly $\dim S_K^{(n)} = nK-1$. A function $\varphi \in H_h^{(n)}[0,X]$ is uniquely defined by its values at the mesh nodes $\varphi_j = \varphi(x_j)$, $j = \overline{0,K}$, with $\varphi_0 = \varphi_K = 0$, and inside the elements $\varphi_{j-1/2} = \{\varphi(x_{j-1} + (l/n)h)\}_{l=1}^{n-1}$, $j = \overline{1,K}$, that form the element in $S_K^{(n)}$.

We utilize the following scaled operator form of the standard FEM discretization for problem (1)

$$\mathcal{A}v = \lambda \mathcal{C} v, \quad v \in S_K^{(n)}, \quad v \neq 0. \tag{2}$$

Here $\mathcal{A} = \mathcal{A}^T > 0$ and $\mathcal{C} = \mathcal{C}^T > 0$ are the global (scaled) stiffness and mass operators (matrices) acting in $S_K^{(n)}$ and together with λ *independent on* h; the true approximate eigenvalues are $\lambda_h = 4h^{-2}\lambda$.

Let $A = \{A_{kl}\}_{k,l=0}^n$ and $C = \{C_{kl}\}_{k,l=0}^n$ be the local stiffness and mass matrices related to the reference element $\sigma_0 = [-1,1]$ with the following entries

$$A_{kl} = \int_{\sigma_0} e_k'(x) e_l'(x)\, dx, \quad C_{kl} = \int_{\sigma_0} e_k(x) e_l(x)\, dx,$$

where $\{e_l\}_{l=0}^n$ is the Lagrange basis in $\mathcal{P}_n$ such that $e_l\big(-1+(2k)/n\big) = \delta_{kl}$, for $k,l = \overline{0,n}$, and δ_{kl} is the Kronecker delta. The matrices A, C and the related matrix pencil have the following 3×3–block form

$$A = \begin{pmatrix} a_0 & a^T & a_n \\ a & \widetilde{A} & \check{a} \\ a_n & \check{a}^T & a_0 \end{pmatrix}, \; C = \begin{pmatrix} c_0 & c^T & c_n \\ c & \widetilde{C} & \check{c} \\ c_n & \check{c}^T & c_0 \end{pmatrix}, \; G(\lambda) := A - \lambda C = \begin{pmatrix} g_0(\lambda) & g^T(\lambda) & g_n(\lambda) \\ g(\lambda) & \widetilde{G}(\lambda) & \check{g}(\lambda) \\ g_n(\lambda) & \check{g}^T(\lambda) & g_0(\lambda) \end{pmatrix}. \tag{3}$$

Here $\widetilde{A}$, $\widetilde{C}$ and $\widetilde{G}(\lambda) = \widetilde{A} - \lambda\widetilde{C}$ are square matrices of order $n-1$ and $a, c, g(\lambda) = a - \lambda c \in \mathbb{R}^{n-1}$ whereas $\check{p}_l \equiv (Pp)_l = p_{n-l}$, $l = \overline{1,n-1}$, for $p \in \mathbb{R}^{n-1}$. Let $\mathbb{R}_e^{n-1}$ and $\mathbb{R}_o^{n-1}$ be the subspaces of even and odd vectors in $\mathbb{R}^{n-1}$, i.e. such that $Pp = p$ and $Pp = -p$. Clearly $p = p_e + p_o$ with $p_e := (p + \check{p})/2$ and $p_o := (p - \check{p})/2$ and thus $\mathbb{R}^{n-1} = \mathbb{R}_e^{n-1} \oplus \mathbb{R}_o^{n-1}$ for $n \geqslant 3$; note that $\mathbb{R}_o^{n-1} = \{0\}$ for $n = 2$.

Then problem (2) can be represented in the following explicit form

$$g_n(\lambda)v_{j-1} + \check{g}(\lambda)\cdot v_{j-1/2} + 2g_0(\lambda)v_j + g(\lambda)\cdot v_{j+1/2} + g_n(\lambda)v_{j+1} = 0, \quad j = \overline{1,K-1},$$
$$g(\lambda)v_{j-1} + \widetilde{G}(\lambda)v_{j-1/2} + \check{g}(\lambda)v_j = 0, \quad j = \overline{1,K},$$

with $v_0 = v_K = 0$, $v \not\equiv 0$; see the similar problem for $\lambda \in \mathbb{C}$ on the uniform mesh on $[0,\infty)$ in [9]. Hereafter the symbol $\cdot$ denotes the inner product of vectors in $\mathbb{R}^{n-1}$.

We also consider the auxiliary eigenvalue problems on and inside the reference element σ_0

$$Ae = \lambda Ce, \quad e \in \mathbb{R}^{n+1}, \quad e \neq 0; \quad \widetilde{A}e = \lambda\widetilde{C}e, \quad e \in \mathbb{R}^{n-1}, \quad e \neq 0, \tag{4}$$

where clearly $A \geqslant 0$, $C > 0$ and $\widetilde{A} = \widetilde{A}^T > 0$, $\widetilde{C} = \widetilde{C}^T > 0$; see some their properties in [9]. Denote by S_n and $\tilde{S}_n$ their spectra. Let $\{\lambda_0^{(l)}, e^{(l)}\}_{l=1}^{n-1}$ be eigenpairs of the second problem (4).

Lemma 2.1 *1. Any eigenvalue $\lambda_0^{(l)}$ is positive and at most double. For simple $\lambda_0^{(l)}$, the corresponding eigenvector $e^{(l)}$ is even or odd; for double $\lambda_0^{(l)} = \lambda_0^{(l+1)}$, we can choose $e^{(l)}$ even and $e^{(l+1)}$ odd; then $\{e^{(l)}\}_{l=1}^{n-1}$ forms the basis in $\mathbb{R}^{n-1}$.*

2. Similar properties are valid for the eigenpairs of the first problem (4) *with the exception of one simple zero eigenvalue.*

One can check by the direct computation that all the eigenvalues in S_n and $\tilde S_n$ are simple at least for $1 \leqslant n \leqslant 9$, see [9]. For low n, one can find S_n and $\tilde S_n$ exactly, in particular, $\tilde S_2 = \{2.5\}$, $\tilde S_3 = \{2.5, 10.5\}$, $\tilde S_4 = \{14 \pm \sqrt{133}, 10.5\}$ and $\tilde S_5 = \{14 \pm \sqrt{133}, 30 \pm 9\sqrt{5}\}$.

We choose $\{e^{(l)}\}_{l=1}^{n-1}$ as in Lemma 2.1 using scaling $\widetilde{C}e^{(l)} \cdot e^{(l)} = 1$.

Lemma 2.2 *Let $\widetilde{G}(\lambda)p = -g(\lambda)$, see (3), where $\lambda \notin \tilde S_n$. Then the following formulas hold*

$$p = \sum\nolimits_{l=1}^{n-1} \frac{a^{(l)} - \lambda c^{(l)}}{\lambda - \lambda_0^{(l)}} e^{(l)} = \sum\nolimits_{l=1}^{n-1} \frac{a^{(l)} - \lambda_0^{(l)} c^{(l)}}{\lambda - \lambda_0^{(l)}} e^{(l)} - \widetilde{C}^{-1} c.$$

Here $\{a^{(l)}\}_{l=1}^{n-1}$ and $\{c^{(l)}\}_{l=1}^{n-1}$ are the expansion coefficients of the vectors a and c, see (3), with respect to the basis $\{\widetilde{C}e^{(l)}\}_{l=1}^{n-1}$, for example, $c = \sum_{l=1}^{n-1} c^{(l)} \widetilde{C} e^{(l)}$ with $c^{(l)} = c \cdot e^{(l)}$.

2. Below we need to assume that all the eigenvalues in both S_n and $\tilde S_n$ are simple for considered n. We introduce the auxiliary equation

$$\widehat{\gamma}(\lambda) \equiv -(g_0 - g \cdot \widetilde{G}^{-1} g)(\lambda) / (g_n - \check{g} \cdot \widetilde{G}^{-1} g)(\lambda) = \theta$$

with the parameter θ, see [9]. Owing to Lemma 2.2 this equation can be rewritten as

$$a_0 - \lambda c_0 + \sum\nolimits_{l=1}^{n-1} \frac{(a^{(l)} - \lambda c^{(l)})^2}{\lambda - \lambda_0^{(l)}} = -\theta \Big(a_n - \lambda c_n + \sum\nolimits_{l=1}^{n-1} \frac{(\check{a}^{(l)} - \lambda \check{c}^{(l)})(a^{(l)} - \lambda c^{(l)})}{\lambda - \lambda_0^{(l)}} \Big). \tag{5}$$

Its solving is equivalent to finding the roots of a polynomial having at most nth degree. Here $\check{a}^{(l)} = \check{a} \cdot e^{(l)}$ and $\check{c}^{(l)} = \check{c} \cdot e^{(l)}$. Moreover, for $2 \leqslant n \leqslant 9$ computations help to confirm that the vectors $e^{(l)}$ are even and odd respectively for odd and even l; therefore $\check{a}^{(l)} = (-1)^l a^{(l)}$ and $\check{c}^{(l)} = (-1)^l c^{(l)}$, $l = \overline{1, n-1}$.

We define the simplest inner product in $S_K^{(n)}$ and the squared $\mathcal{C}$-norm

$$(y, v)_{S_K^{(n)}} := \sum\nolimits_{j=1}^{K-1} y_j v_j + \sum\nolimits_{j=1}^{K} y_{j-1/2} \cdot v_{j-1/2}, \quad \|v\|_{\mathcal{C}}^2 := (\mathcal{C}v, v)_{S_K^{(n)}}.$$

Next theorem presents eigenvalues and eigenvectors of problem (2).

Theorem 2.3 *1. The spectrum of problem (2) consists in $\tilde S_n$ and the numbers $\{\lambda_k^{(l)}\}_{l=1}^{n} \not\subset \tilde S_n$ that are all n (and all positive real) solutions to equation (5) with $\theta = \theta_k := \cos \frac{\pi k}{K}$ for $k = \overline{1, K-1}$ and are different for fixed k.*

2. To the eigenvalue $\lambda_0^{(l)}$, the following eigenvector corresponds

$$s_{0,j}^{(l)} = 0, \;\; j = \overline{1, K-1}, \quad s_{0,j-1/2}^{(l)} = (-P)^{j-1} e^{(l)}, \;\; j = \overline{1, K},$$

for $l = \overline{1, n-1}$. Here $(-P)^{j-1} e = (-1)^{j-1} e$ for even e, $(-P)^{j-1} e = e$ for odd e.

3. To the eigenvalue $\lambda_k^{(l)}$, the following eigenvector corresponds

$$s_{k,j}^{(l)} = \sin \frac{\pi k j}{K}, \;\; j = \overline{1, K-1}, \quad s_{k,j-1/2}^{(l)} = p_k^{(l)} \sin \frac{\pi k (j-1)}{K} + \check{p}_k^{(l)} \sin \frac{\pi k j}{K}, \;\; j = \overline{1, K},$$

where $p_k^{(l)} \in \mathbb{R}^{n-1}$ is the solution to non-degenerate algebraic system $\widetilde{G}\big(\lambda_k^{(l)}\big) p_k^{(l)} = -g\big(\lambda_k^{(l)}\big)$, for $k = \overline{1, K-1}$, $l = \overline{1, n}$.

4. The introduced eigenvectors are C-orthogonal, i.e. $(C s_k^{(l)}, s_{\tilde k}^{(\tilde l)})_{S_K^{(n)}} = 0$ for any $k, \tilde k \in \overline{0, K-1}$, $l \in \overline{1, n - \delta_{k0}}$ and $\tilde l \in \overline{1, n - \delta_{\tilde k 0}}$ such that $k \neq \tilde k$ and/or $l \neq \tilde l$.

They form the basis in $S_K^{(n)}$, i.e. any $w \in S_K^{(n)}$ can be uniquely expanded as

$$w = \sum\nolimits_{l=1}^{n-1} w_{0l} s_0^{(l)} + \sum\nolimits_{k=1}^{K-1} \sum\nolimits_{l=1}^{n} w_{kl} s_k^{(l)}. \tag{6}$$

Notice that: (1) the vectors $s_0^{(l)}$ are used only to describe the algorithm, and only the vectors $e^{(l)}$ are applied in its implementation; (2) $s_{k,j}^{(l)}$ are independent on l; (3) the vectors $p_k^{(l)}$ can also be computed owing to Lemma 2.2.

3. We call the calculation of $w \in S_K^{(n)}$ by the coefficients w_{kl} of the expansion (6) as *the inverse F_n-transform* and the calculation of the coefficients w_{kl} by $w \in S_K^{(n)}$ as *the direct F_n-transform*. Let us describe their fast FFT-based implementation.

Theorem 2.4 *1. The inverse F_n-transform can be implemented according to the following formulas*

$$w_j = \sum\nolimits_{k=1}^{K-1} \Big(\sum\nolimits_{l=1}^{n} w_{kl} \Big) \sin \frac{\pi k j}{K}, \quad j = \overline{1, K-1},$$

$$w_{j-1/2} = (-P)^{j-1} \sum\nolimits_{l=1}^{n-1} w_{0l} e^{(l)}$$

$$+2 \sum\nolimits_{k=1}^{K-1} d_{k,e} \cos \frac{\pi k}{2K} \sin \frac{\pi k (j-1/2)}{K} - 2 \sum\nolimits_{k=1}^{K-1} d_{k,o} \sin \frac{\pi k}{2K} \cos \frac{\pi k (j-1/2)}{K}, \quad j = \overline{1, K},$$

where $d_{k,e}$ and $d_{k,o}$ are respectively even and odd components of the vectors $d_k := \sum_{l=1}^{n} w_{kl} p_k^{(l)}$. Note that $(-P)^{j-1} e = e$ for odd j and $(-P)^{j-1} e = -\check{e}$ for even j for any $e \in \mathbb{R}^{n-1}$.

The collection $\{w_j\}_{j=1}^{K-1}$ can be computed by the standard inverse FFT with respect to sines. The collection $\{w_{j-1/2}\}_{j=1}^{K}$ can be computed by $n-1$ modified inverse FFT related to the centers of elements in the amount of $[n/2]$ with respect to sines and $[(n-1)/2]$ with respect to cosines using extensions $d_{K,e} := 0$ and $d_{0,o} := 0$, see algorithms DST-I, DST-III and DCT-III in [2].

2. The direct F_n-transform can be implemented starting from the standard formulas

$$w_{kl} = (\mathcal{C} w, s_k^{(l)})_{S_K^{(n)}} / \|s_k^{(l)}\|_{\mathcal{C}}^2.$$

Here, first, for $k = 0$, $l = \overline{1, n-1}$, we have

$$(\mathcal{C} w, s_0^{(l)})_{S_K^{(n)}} = \Big(\widetilde{C} \sum\nolimits_{j=1}^{K} (-P)^{j-1} w_{j-1/2} \Big) \cdot e^{(l)}, \quad \|s_0^{(l)}\|_{\mathcal{C}}^2 = K.$$

Second, for $k = \overline{1, K-1}$, $l = \overline{1, n}$ and $y := Cw$, we have

$$(y, s_k^{(l)})_{S_K^{(n)}} = \sum\nolimits_{j=1}^{K-1} y_j \sin \frac{\pi k j}{K}$$

$$+ p_{k,e}^{(l)} \cdot \sum\nolimits_{j=1}^{K-1} (y_{j-1/2} + y_{j+1/2})_e \sin \frac{\pi k j}{K} + p_{k,o}^{(l)} \cdot \sum\nolimits_{j=1}^{K-1} (y_{j+1/2} - y_{j-1/2})_o \sin \frac{\pi k j}{K},$$

$$\|s_k^{(l)}\|_{\mathcal{C}}^2 = K(b_{kl,0} + b_{kl,n} \theta_k), \;\; b_{kl,0} = c_0 + (\widetilde{C} p_k^{(l)} + 2c) \cdot p_k^{(l)}, \;\; b_{kl,n} = c_n + (\widetilde{C} p_k^{(l)} + 2c) \cdot \check{p}_k^{(l)}.$$

The collection of all these coefficients can be computed using n standard direct FFTs with respect to sines.

4. Now we consider in detail solving of the N-dimensional boundary value problem

$$-\Delta u + \alpha u = f \;\; \text{в} \;\; \Omega = (0, X_1) \times \ldots \times (0, X_N), \quad u|_{\partial\Omega} = 0, \tag{7}$$

where Δ is the Laplace operator and $\alpha = \text{const}$; for simplicity, let $\alpha > -\pi^2 \big(X_1^{-2} + \ldots + X_N^{-2} \big)$.

We introduce the space $H_{h_1}^{(n_1)}[0, X_1] \otimes \ldots \otimes H_{h_N}^{(n_N)}[0, X_N]$ of the piecewise-polynomial in $\overline{\Omega}$ functions, where $h_i = X_i / K_i$ and $n_i \geqslant 2$, $i = \overline{1, N}$. Let $\mathbf{K} = (K_1, \ldots, K_N)$ and $\mathbf{n} = (n_1, \ldots, n_N)$.

We define the space $S_{\mathbf{K}}^{(\mathbf{n})} = S_{K_1}^{(n_1)} \otimes \ldots \otimes S_{K_N}^{(n_N)}$ of vector functions. Similarly to the 1D case, there is the natural isomorphism between functions in $H_{h_1}^{(n_1)}[0, X_1] \otimes \ldots \otimes H_{h_N}^{(n_N)}[0, X_N]$ and vectors in $S_{\mathbf{K}}^{(\mathbf{n})}$.

The FEM dicretization of problem (7) can be written in the following operator form

$$(4h_1^{-2}\mathcal{A}_1\mathcal{C}_2\dots\mathcal{C}_N+\dots+4h_N^{-2}\mathcal{A}_N\mathcal{C}_1\dots\mathcal{C}_{N-1})v+\alpha\mathcal{C}_1\dots\mathcal{C}_Nv=f^h,\quad v\in S^{(\mathbf{n})}_{\mathbf{K}}, \tag{8}$$

where $\mathcal{A}_i$ and $\mathcal{C}_i$ are versions of the above defined operators $\mathcal{A}$ and $\mathcal{C}$ acting in variable x_i (depending on K_i and n_i), $i=\overline{1,N}$, and $f^h\in S^{(\mathbf{n})}_{\mathbf{K}}$ is the FEM average of f. Remind that the general case $u|_{\partial\Omega}=b$ in (7) could be covered by reducing to (8) with the modified f^h depending on b^h (the FEM average of b).

To compute its solution, the F_n-transforms from Theorem 2.4 can be applied twofold.

(a) Let the vector $\varphi^h\in S^{(\mathbf{n})}_{\mathbf{K}}$ be the solution to the auxiliary algebraic problem $\mathcal{C}_1\dots\mathcal{C}_N\varphi^h=f^h$ with the splitting operator (the product of operators acting in $x_1,\dots,x_N$), i.e. formally $\varphi^h=\mathcal{C}_1^{-1}\dots\mathcal{C}_N^{-1}f^h$. We consider the multiple expansion of $\varphi^h\in S^{(\mathbf{n})}_{\mathbf{K}}$ like (6)

$$\varphi^h=\sum\nolimits_{i=1}^{N}\sum\nolimits_{k_i=0}^{K_i-1}\sum\nolimits_{l_i=1}^{n_i-\delta_{k_i0}}\varphi^h_{k_1l_1,\dots,k_Nl_N}s^{(l_1)}_{1,\,k_1}\dots s^{(l_N)}_{N,\,k_N}. \tag{9}$$

Then the expansion of the solution has the following form

$$v=\sum\nolimits_{i=1}^{N}\sum\nolimits_{k_i=0}^{K_i-1}\sum\nolimits_{l_i=1}^{n_i-\delta_{k_i0}}\frac{\varphi^h_{k_1l_1,\dots,k_Nl_N}}{4h_1^{-2}\lambda^{(l_1)}_{1,\,k_1}+\dots+4h_N^{-2}\lambda^{(l_N)}_{m,\,k_N}+\alpha}s^{(l_1)}_{1,\,k_1}\dots s^{(l_N)}_{N,\,k_N}. \tag{10}$$

Here $\left\{\lambda^{(l_i)}_{i,k_i},s^{(l_i)}_{i,k_i}\right\}$ are versions of the above defined eigenpairs $\left\{\lambda^{(l)}_k,s^{(l)}_k\right\}$ with respect to x_i.

The steps of the algorithm (a) are rather standard:

(1) solving the auxiliary problem $\mathcal{C}_1\dots\mathcal{C}_N\varphi^h=f^h$ for φ^h (that is reduced to the sequential solving of the 1D problems in x_1 with the matrix $\mathcal{C}_1$,..., x_N with the matrix $\mathcal{C}_N$);

(2) finding the coefficients of expansion (9) for φ^h (by the direct F_n-transforms in x_1,..., x_N);

(3) finding v by the coefficients of its expansion (10) (by the inverse F_n-transforms in x_1,..., x_N).

(b) Let the vector $\varphi^h\in S^{(\mathbf{n})}_{\mathbf{K}}$ be the solution to the auxiliary $(m-1)$D problem $\mathcal{C}_2\dots\mathcal{C}_N\varphi^h=f^h$ in x_2,..., x_N, i.e. formally $\varphi^h=\mathcal{C}_2^{-1}\dots\mathcal{C}_N^{-1}f^h$. We consider the expansion of φ^h like (6) in x_2,..., x_N, i.e.

$$\varphi^h=\sum\nolimits_{i=2}^{N}\sum\nolimits_{k_i=0}^{K_i-1}\sum\nolimits_{l_i=1}^{n_i-\delta_{k_i0}}\varphi^h_{k_2l_2,\dots,k_Nl_N}s^{(l_2)}_{2,\,k_2}\dots s^{(l_N)}_{N,\,k_N}, \tag{11}$$

now with the coefficients $\varphi^h_{k_2l_2,\dots,k_Nl_N}\in S^{(n_1)}_{K_1}$. Then the coefficients $v_{kl}\in S^{(n_1)}_{K_1}$ in the similar expansion of the solution $v\in S^{(\mathbf{n})}_{\mathbf{K}}$

$$v=\sum\nolimits_{i=2}^{N}\sum\nolimits_{k_i=0}^{K_i-1}\sum\nolimits_{l_i=1}^{n_i-\delta_{k_i0}}v_{k_2l_2,\dots,k_Nl_N}s^{(l_2)}_{2,\,k_2}\dots s^{(l_N)}_{N,\,k_N}, \tag{12}$$

serve as the solutions to 1D problems in x_1

$$\left[4h_1^{-2}\mathcal{A}_1+(4h_2^{-2}\lambda^{(l_2)}_{k_2}+\dots+4h_N^{-2}\lambda^{(l_N)}_{k_N}+\alpha)\mathcal{C}_1\right]v_{k_2l_2,\dots,k_Nl_N}=\varphi^h_{k_2l_2,\dots,k_Nl_N}. \tag{13}$$

Their matrices are symmetric and positive definite. Of course, the simpler case $n_1=1$ is acceptable too.

The steps of the algorithm (b) are rather standard as well:

(1) solving the auxiliary problem $\mathcal{C}_2\dots\mathcal{C}_N\varphi^h=f^h$ for φ^h;

(2) finding the coefficients of the expansion (11) for φ^h (by the direct F_n-transforms in x_2,..., x_N);

(3) solving the collection of the 1D problems (13) for the coefficients of the expansion of v;

(4) finding v by the coefficients of its expansion (12) (by the inverse F_n-transforms in x_2,..., x_N).

κ	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$
3	$5.4E^{-3}$	$1.9E^{-5}$	$8.6E^{-6}$	$5.0E^{-7}$	$8.2E^{-9}$	$2.0E^{-10}$	$1.9E^{-12}$	$3.5E^{-14}$	$1.2E^{-14}$
4	$1.4E^{-3}$	$1.2E^{-6}$	$5.4E^{-7}$	$1.6E^{-8}$	$1.3E^{-10}$	$1.6E^{-12}$	$8.5E^{-15}$	$1.0E^{-14}$	$4.5E^{-14}$
5	$3.5E^{-4}$	$7.7E^{-8}$	$3.4E^{-8}$	$5.1E^{-10}$	$1.9E^{-12}$	$1.3E^{-14}$	$9.3E^{-15}$	$1.3E^{-14}$	$6.4E^{-14}$
6	$8.7E^{-5}$	$4.8E^{-9}$	$2.1E^{-9}$	$1.6E^{-11}$	$3.0E^{-14}$	$6.1E^{-16}$	$5.6E^{-15}$	$5.2E^{-15}$	$1.7E^{-14}$
7	$2.2E^{-5}$	$3.0E^{-10}$	$1.3E^{-10}$	$4.9E^{-13}$	$2.4E^{-15}$	$9.4E^{-16}$	$6.4E^{-15}$	$7.4E^{-15}$	$4.5E^{-15}$
8	$5.4E^{-6}$	$1.9E^{-11}$	$8.3E^{-12}$	$1.6E^{-14}$	$1.9E^{-15}$	$1.6E^{-15}$	$7.3E^{-15}$	$9.2E^{-15}$	$1.6E^{-14}$
9	$1.4E^{-6}$	$1.2E^{-12}$	$5.2E^{-13}$	$6.9E^{-16}$	$1.3E^{-15}$	$1.4E^{-15}$	$2.7E^{-15}$	$8.5E^{-15}$	$2.8E^{-14}$
10	$3.4E^{-7}$	$7.4E^{-14}$	$3.3E^{-14}$	$6.1E^{-16}$	$6.9E^{-16}$	$1.7E^{-15}$	$2.2E^{-15}$	$1.6E^{-14}$	$3.4E^{-14}$

Table 1: Errors in the uniform norm in dependence on $K = 2^\kappa = 8, 16, ..., 1024$ and $n = \overline{1,9}$

Implementing algorithms (a) and (b) costs respectively $O(K \log_2 K)$ and $O\big(K \log_2(K_2 \dots K_N)\big)$ arithmetic operations with $K = K_1 \dots K_N$.

They can be applied to solve various time-dependent PDEs such as the heat, wave or Schrödinger's equations since usually their implicit time discretizations lead to problems like (8) at the upper time level.

Moreover, algorithm (b) is directly extended to the case of more general equations than in (7) with the coefficients depending on x_1, various boundary conditions for $x_1 = 0, X_1$ and the nonuniform mesh in x_1 [7]. It can also be applied to reduce 3D problems in a cylindrical domain to a collection of independent 2D problems in the cylinder base.

5. Both algorithms (a) and (b) are well-behaved in the numerical experiments. We choose problem (7) for $N = 2$, $\alpha = 1$ and $X_1 = X_2 = 1$, with the exact solution $u(x_1, x_2) := \sin(\pi x_1)\sin(\pi x_2)(x_1+x_2-1)$ and take $K_1 = K_2 = K$ and $n_1 = n_2 = n$. The errors for algorithm (a) in the uniform norm are given in Table 1 in dependence on $K = 8, 16, ..., 1024$, for $n = \overline{1,9}$. We emphasize that there is almost no impact of the round-off errors as K and n grows. Here the multiple Gauss quadrature formulas with $n+1$ nodes in x_1 and x_2 were applied to compute f^h, and the eigenvalues of the 1D problems were computed with the quadruple precision (using Mathematica) to improve the stability with respect to round-off errors.

In Fig. 1 we present the execution time for the same K and n, using our codes in Matlab R2016a for both algorithms. The ordinary laptop with Intel Core i3-2350M CPU 2.3 GHz, 4 Gb, Win 7 x64 on board was applied. Including the case $n = 1$ allows us to compare the original well-known algorithms with the above suggested new algorithms for higher n. Notice the rather close to linear behavior of time in K and its mild monotone growth in n. Specify that system (8) contains $(Kn-1)^2$ unknowns. For $K = 1024$ and $n = 9$, this is almost $85 \cdot 10^6$ unknowns but only less than 2 min is required for solving.

Acknowledgement. The study has been funded by the RFBR, grant № 16-01-00048.

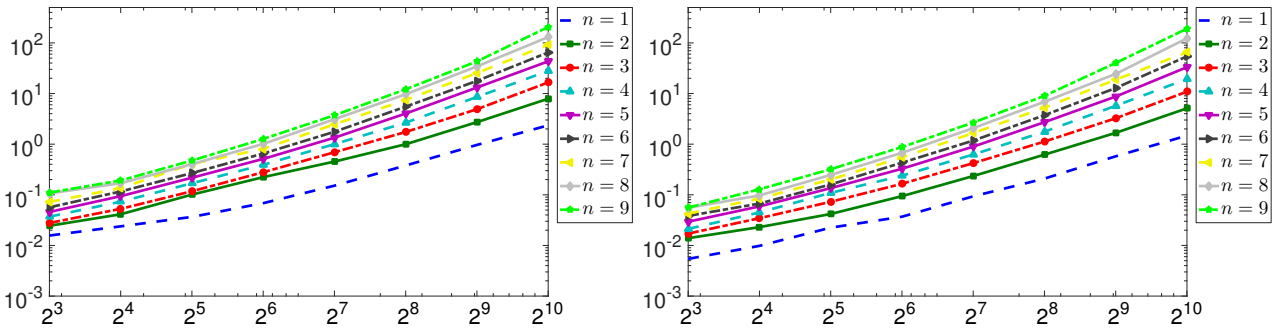

Figure 1: The execution time (in seconds) for algorithms (a) (left) and (b) (right)

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