Algebras of Linear Growth and the Dynamical Mordell–Lang Conjecture

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Spa, 2017

Graded algebras

We call an associative algebra A graded if

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \dots,$$

where $A_0 = k$ is a basic field, $\dim {}_k A_i < \infty$. All our algebras are graded.

Hilbert function: $h_A(n) = \dim A_n$

Hilbert series:

$H_A(z) = \sum_{n \geq 0} z^n \dim A_n = \sum_{n \geq 0} z^n h_A(n).$

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Rationality

An algebra A is called finitely presented if it is defined by a finite number of generators and relations.

Theorem (Govorov, 1972)

If the relations of a finitely presented algebra A are monomials in generators then $H_A(z)$ is a rational function.

Conjecture (Govorov)

For each finitely presented algebra A the Hilbert series $H_A(z)$ is a rational function.

First counterexamples: Ufnarovski, 1978 and Shearer, 1980.
Open questions: Govorov conjecture for Noetherian algebras and for Koszul algebras.

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Rationality in the linear growth case

An algebra has linear growth, if GK–dim $A \leq 1$, that is, for some $c > 0$ we have $h_A(n) = \dim A_n < c$.

Example

Let $A = \langle x, y | x^2, yxy, xy^{2^t}x \text{ for all } t \geq 0 \rangle$. Then $A_n = k\{y^n, xy^{n-1}, y^{n-1}x, xy^{n-2}x\}$ for $n \neq 2^t + 2$ or $A_n = k\{y^n, xy^{n-1}, y^{n-1}x\}$ otherwise.
We have $H_A(z) = 1 + 2z + 4z^2/(1-z) - z^2 \sum_{t \geq 0} z^{2^t}$.

Problem (Govorov conjecture for algebras of linear growth, GALG)

Suppose that an algebra A of linear growth is finitely presented. Is $H_A(z)$ a rational function?

For such algebras, $H_A(z)$ is rational iff $h_A(n)$ is eventually periodic, that is, $\exists n_0, T > 0$ such that $h_A(n) = h_A(n+T)$ for all $n > n_0$.

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Automaton algebras

Let X be a finite generating set of an algebra A. Consider a multiplicative ordering '$<$' of the set of all words in X. A word on X is called normal in A if it is not a linear combination of less words. The set N of all normal words is a linear basis of A.

Definition (Ufnarovski)

An algebra A is called automaton if N is a regular language.

Recall that a language is regular iff it is recognized by a finite automaton.

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Property (generalized Govorov theorem): *If A is graded automaton (that is, it is graded and automaton with homogeneous X and a degree-compatible ordering '$<$'), then $H_A(z)$ is a rational function.*

Conjecture (Ufnarovski, 1990)

A finitely presented algebra of linear growth is automaton.

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UGA implies GALG.

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The finite characteristic case

Theorem

Suppose that the field k has a finite characteristic. Then both Govorov conjecture for algebra of linear growth and Ufnarovski conjecture for graded algebras hold if and only if k is an algebraic extension of its prime subfield.

'If' part: essentially, the case of finite field.
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The case of infinite field

What about the case char $k = 0$?

Example (Fermat algebras)

For $\alpha, \beta \in k^\times$, let $A = A_{\alpha,\beta}$ be generated by a, b, c, x, y, z subject to 26 relations $xc - \alpha cx, yb - \beta cy$ and others. Then $h_A(n+3)$ is 10 or 11 according to whether the Fermat equality $\alpha^n + \beta^n = 1$ holds. So, it has no nonzero solution in $k^\times$ for each $n \geq 3$ if and only if $h_A(i) = 10$ for all $i \geq 6$ and each $A = A_{\alpha,\beta}$.

Theorem

Let $g \geq 5$ be an integer. If the field k is infinite, then there are infinitely many (periodic) sequences h_A for g-generated quadratic k-algebras of linear growth. If If, in addition, k contains all primitive roots of unity, then both the length d of the initial non-periodic segment and the period T of h_A can be arbitrary large.

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Skolem–Mahler–Lech theorem

Theorem (Skolem–Mahler–Lech)

If $\operatorname{char} k = 0$ *and* $a_n = c_1 a_{n-1} + \cdots + c_d a_{n-d}$ *is a linear recurrence over* k*, then the zero set* $\{n \geq 0 | a_n = 0\}$ *is the finite union of several arithmetic progressions and a finite set.*

We refer to a set of nonnegative integers which is the union of a finite set and a finite collection of arithmetic progressions as SML.
All known proofs essentially use p-adic analysis.

Example (Lech, 1953)

If the field k has characteristic $p > 0$ and $t \in k$ is transcendental over the prime subfield F_p then the sequence $a_n = (t+1)^n - t^n - 1$ satisfies $a_n = 0$ iff $n = p^m$ with $m \geq 0$.

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The dynamical Mordell–Lang conjecture: formulation

A challenging generalization:

Conjecture (The dynamical Mordell–Lang conjecture)

Let $\mathcal{V}$ be a quasiprojective variety over k (of characteristic zero), let $\Phi : \mathcal{V} \to \mathcal{V}$ be any morphism, and let $\alpha \in \mathcal{V}$. Then for each subvariety $Y \subset \mathcal{V}$, the set $\{n \geq 0 | \Phi^n(\alpha) \in Y\}$ is SML.

See the book (which is unfortunately absent in our library...) :Jason P. Bell, Dragos Ghioca, Thomas J. Tucker, *The Dynamical Mordell-lang Conjecture*, AMS, 2016 (Mathematical Surveys and Monographs, vol. 210)

The dynamical Mordell–Lang conjecture: some results

Some important cases are known.

Theorem (Bell, 2006)

Let $\operatorname{char} k = 0$. The conjecture is true provided that $\mathcal{V}$ is an affine variety and Φ is a polynomial automorphism.

Note that this corollary implies the Skolem–Mahler–Lech theorem.

Theorem

For any field k, GALG implies the linear dynamical Mordell–Lang conjecture.

So, we have implications
UGA$\Longrightarrow$GALG$\Longrightarrow$Linear ML$\Longrightarrow$ Skolem–Mahler–Lech.

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Generalized dynamical Mordell–Lang conjecture

Does dML implies UGA or (at least) GALG?

Let A is generated and related in degrees at most D. Let $V = \text{mod-}A^{\leq D}$ be the category of graded modules with generators and relations in degrees $0..D$. Then $F : M \mapsto M_{\geq 1}[1]$ is an endofunctor of V. For $m > 0$, let $V_m = \{M \in \mathcal{O}b\, V | \dim M_0 = m\}$.

Then GALG is equivalent to the following categorical version of dML: *Given a finitely presented graded algebra A of linear growth as above and a positive integer m, is the set $\{n \geq 0 | F^n(A) \in V_m\}$ always SML?*

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Thank you!

Proof of the main theorem

'Only if' k an algebraic extension of $F_p \Longrightarrow$ UGA $\Longrightarrow$ GALG

For example, let us show: k is finite $\Longrightarrow$ GALG

Proof. There are finite number (say, N) of isomorphism classes of $M \in \text{mod-}A^{\leq D}$ such that $\dim M_0 \leq C$ (where $\dim A_n \leq C$ for all $n \geq 0$). Then the sequence $h_A(n) = \dim F^n(A)_0$ is periodic with both period and non-periodic segment of length at most N.

'If'

k contains $F_p[t] \Longrightarrow$ GALG fails.

Using Lech example ($a_n = (t+1)^n - t^n - 1$): there is a 6-generated algebra with generators a, b, c, x, y, z with relations $xc - (t+1)cx, yc - tcy, zc - cz$ and 23 others (the same as for Fermat algebras). Then

$$h_A(n+3) = \begin{cases} 11, & n = p^m \text{ for some } m \geq 0, \\ 10, & \text{otherwise.} \end{cases}$$

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$$h_A(n+3) = \begin{cases} 11, & n = p^m \text{ for some } m \geq 0, \\ 10, & \text{otherwise.} \end{cases}$$

Normal words in algebras of linear growth

The next Proposition follows from [Belov and others, 1997].

Proposition

If a (non-graded) algebra A of linear growth is generated by a finite set S, then there are $U, V, W \subset S^*$ such that each normal word in A has the form

$$w = ac^n b, \text{ where } a \in U, b \in V, c \in W, n \geq 0.$$

Proposition

Suppose a language L consists of the words of the above form. Then L is regular iff for each $a \in U, b \in V, c \in W$ the set

$$N_{a,b,c} = \{n \in \mathbb{Z}_+ | ac^n b \in L\}$$

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Normal words in algebras of linear growth

Corollary

Suppose that the algebra A is automaton. Then there are a generating set $1 \in S \subset A$ and an ordering such that for some $Q \subset S^3$ the set of normal words in A is

$$\{ac^n b | n \geq 0, (a, b, c) \in Q\}.$$

Thank you very much!