Alexander D. Smirnov
Professor, Dr. Econ. Sci

Stochastic Linear-Logistic Dynamics of Macrofinancial Capital Ratio and Leverage

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Literature

- **Leverage and Financial Instability**
- *Voprosy Economiki*, #9, 1-27, 2012 (in Russian)
-
- **Logistic Model of Financial Leverage**
- *HSE Economic Journal,vol.17, #4,*585-616, 2013 (in Russian)
-
- **Financial Assets Collateralization and Stochastic Leverage**
- *HSE Economic Journal,vol.18, #2,* 183-215, 2014 (in Russian)
-
- **Stochastic Leverage of the Global Financial System,**
- *Proceedings of XVI International April Conference*, NRU HSE, 732-741, Moscow, 2016
-
- **Stochastic Logistic Model of the Global Financial Leverage**,
- *The BE Journal of Theoretical Economics*, 2018, issue 1

The Global Financial System and its Structure

Years	Assets, \$ trln	Debt, \$ trln	Capital, \$ trln	Capital ratio, w_t	Leverage, l_t	Spread1, $a_t = \mu_t - r_t$	Spread2, $c_t = \rho_t - r_t$	Parameter, $b_t = a_t^2 / c_t$
2003	123.8	92.6	31.2	0.252	3.97			
2004	144.7	107.5	37.2	0.257	3.89	0.0207	0.0852	0.005
2005	151.8	114.6	37.2	0.245	4.08	-0.0169	-0.066	-0.0043
2006	190.4	149.6	50.8	0.267	3.74	0.0363	0.1474	0.0088
2007	229.7	164.6	65.1	0.283	3.53	0.0273	0.1024	0.0072
2008	214.4	180.9	33.5	0.156	6.4	-0.1656	-0.5844	-0.0468
2009	232.2	185.0	47.2	0.203	4.92	0.0598	0.3858	0.0093
2010	250.1	195.0	55.1	0.220	4.54	0.0242	0.1145	0.0051
2011	255.9	208.8	47.1	0.184	5.43	-0.0481	-0.2165	-0.0107
2012	268.6	216.1	52.5	0.195	5.12	0.0146	0.0797	0.0027
2013	282.8	220.2	62.6	0.221	4.52	0.033	0.1734	0.0063
2014	294.0	225.0	69.0	0.235	4.26	0.018	0.08	0.004

Current financial system is bloated

Global financial assets were about 294 trillion dollars in 2014; Total assets more than 76 percent consist of bank loans and debt securities, were 3.7 times larger than the world GDP. After the "credit crunch" of 2007-2008 the major central banks pumped about 12 trillion of dollars into the world economy overfilling the monetary subsystem with "instantaneous debts", or money.

Was all that huge mountain of debt necessary for the effective functioning of the world economy?

Duality of collateral ratios and interest rates

Every loan or any other debt instrument, implying its ultimate reimbursement[1], requires two types of economic clauses to be effective: collateral ratios and rates of return; otherwise the process of its repayment would be jeopardized.

The idea of duality of interest rates and collateral ratios is one of the basics of economic theory, and in practice it was well understood ages ago.

Duality of collateral ratios and interest rates

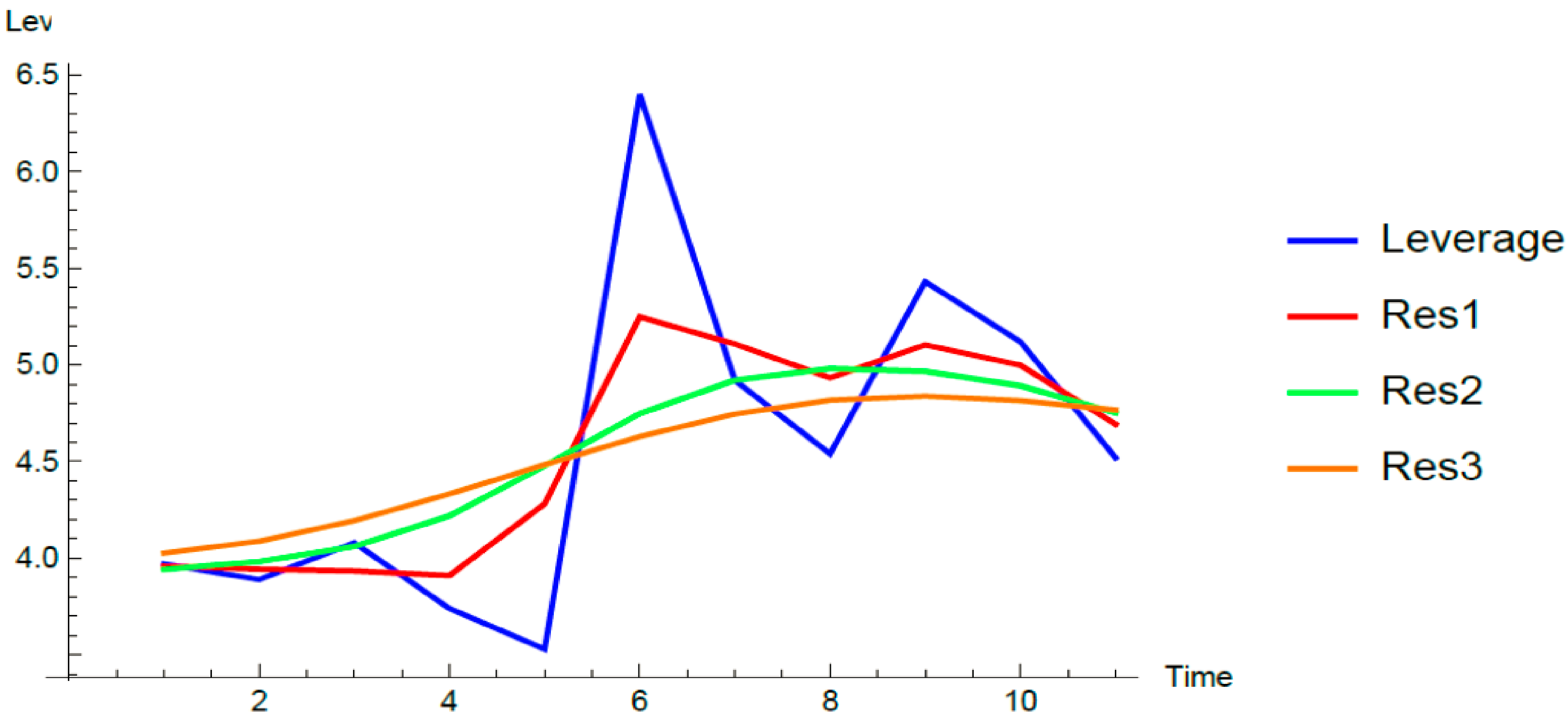
In his immortal masterpiece, *The Merchant of Venice*, W. Shakespeare described the duality of interest rates and collateral ratios precisely:
a loan extended by Shylock to Antonio was interest-free while its redemption was guaranteed by a pound of Antonio's flesh.

Yet the modern science started to study interrelations between collateral ratios and rates of return rather recently, and J. Geanokoplos (1999) was, probably, the first who investigated them simultaneously within a static Arrow-Debreau approach.

Gaussian Filter for Leverage

Gaussian filters with impulse response: $g(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp[-\frac{x^2}{2\sigma^2}]$

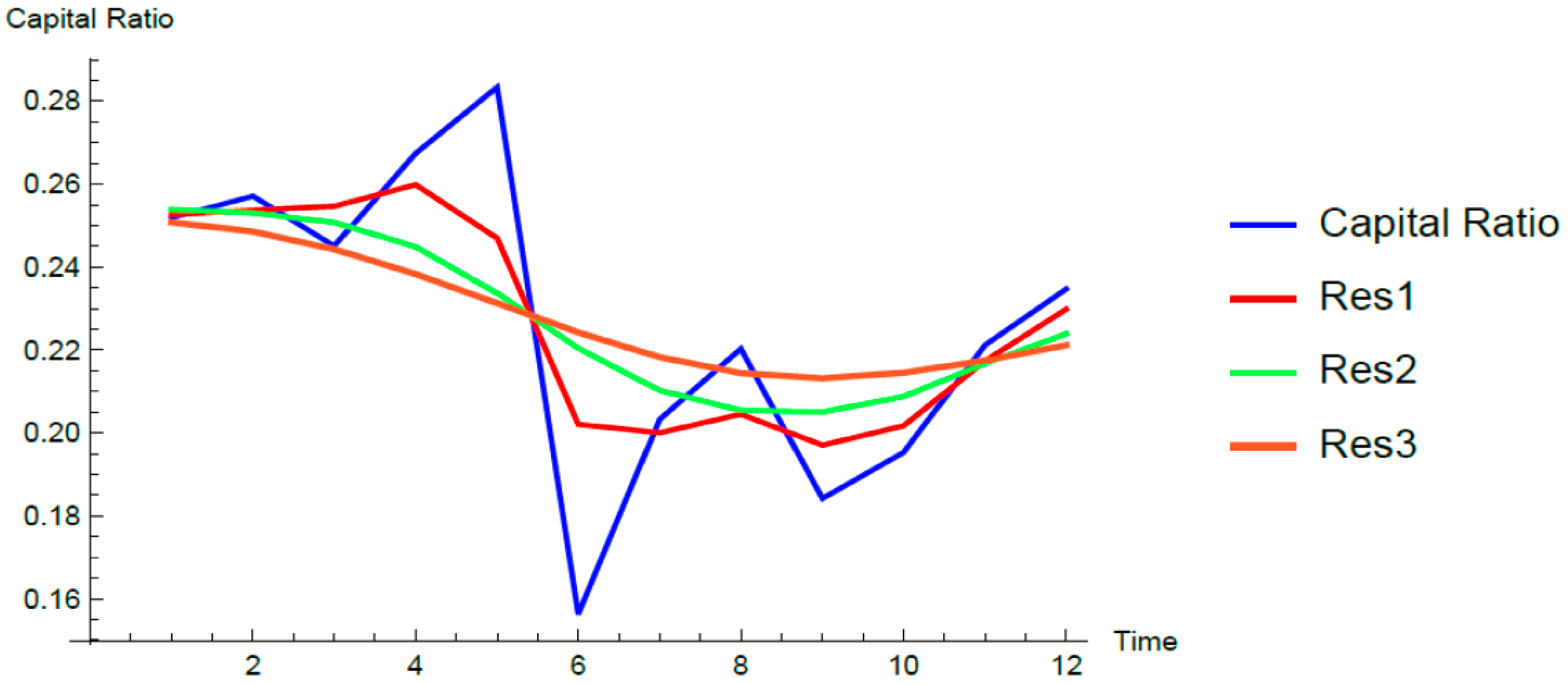
where $x = x(t_i); i = \overline{1, n}$ stands for empirical capital ratio or leverage.



Gaussian Filter for the Capital Ratio

Gaussian filters with impulse response: $g(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp[-\frac{x^2}{2\sigma^2}]$

where $x = x(t_i); i = \overline{1,n}$ stands for empirical capital ratio or leverage.



Linear logistic framework

Gaussian filters revealed nonlinearities in both macrofinancial indicators but at scales that differed by an order of magnitude from each other.

The conceptual ambiguity could be avoided if smaller by order capital ratio is well approximated by a linear process. For mutually coupled processes the linear capital ratio implied the (logistic) nonlinearity of the leverage dynamic, and vice versa.

Descriptors of the long term dynamics

Under some economic assumptions deterministic ODEs of capital ratio and leverage were transformed into appropriate SDEs;
the stationary Kolmogorov-Fokker-Plank equations were solved with regard to the probability density functions;
PDFs were used as general descriptors of the long term capital ratio and leverage dynamics;
Stationary PDFs location around the most probable capital ratio and leverage, and (elementary) probabilities were explained by different market configurations attached to them.

Macrofinancial ratios

The aggregate financial system:

$$A(t) = x(t) + E(t)$$

where $A(t)$ is a value of total assets;
$x(t)$ is a value of total debt;
$E(t)$ is a value of total capital.

There are four basic ratios widely used in analysis and practice:
the collateral ratio, $cr(t) = A(t) / x(t)$;
the loan-to-value ratio, $z(t) = x(t) / A(t)$;
the margin (haircut) ratio, $w(t) = E(t) / A(t)$; and
the leverage, $l(t) = A(t) / E(t)$.

$$cr(t) * z(t) = 1; \quad z(t) + w(t) = 1; \quad w(t) * l(t) = 1.$$

Collateral ratios and leverage

The collateral ratio and the leverage are qualitatively different variables; $z = 0.8$; $w = 0.2$
Their respective series expansions:

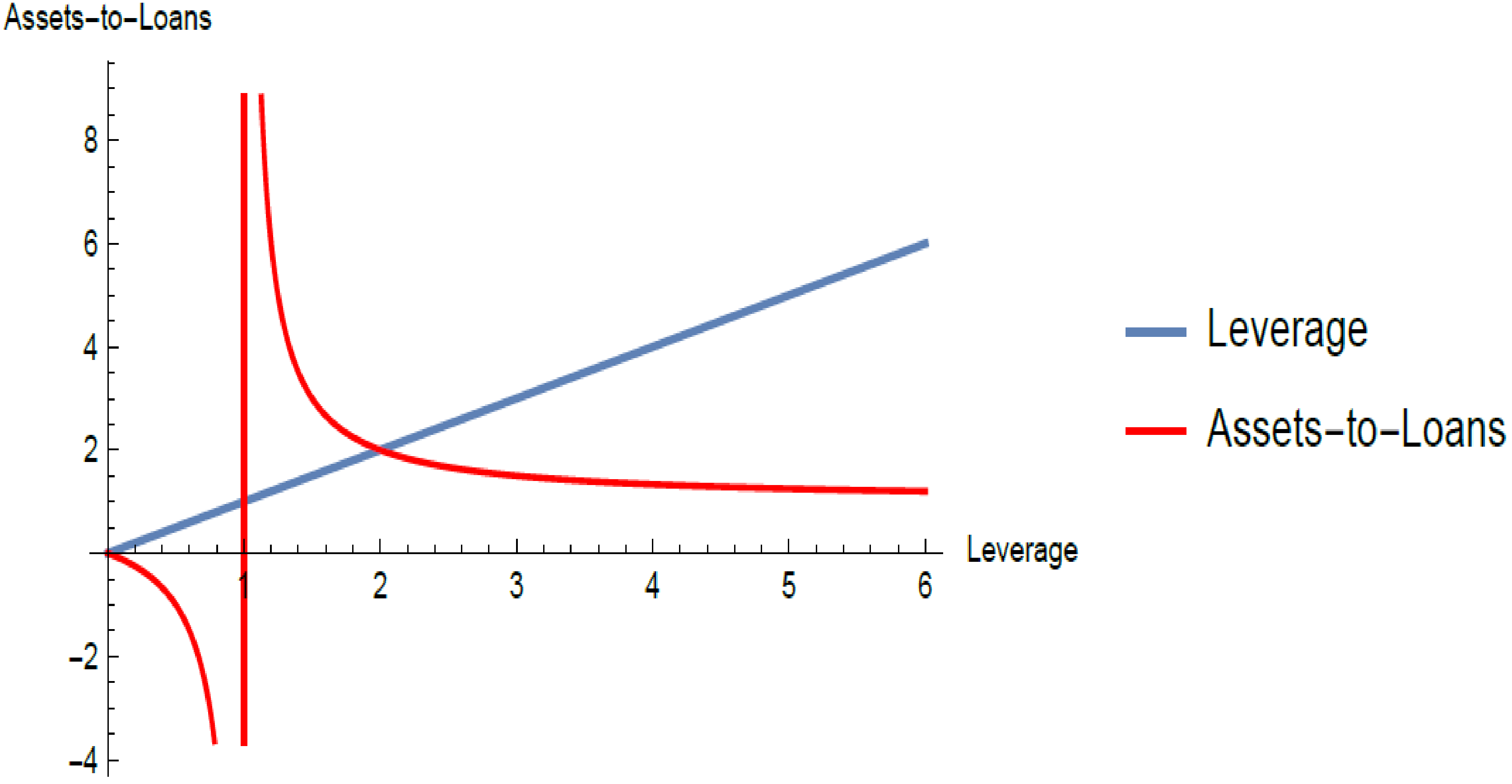
$$cr = (1-w)^{-1} = \sum_{j=0}^{\infty} w^j; \quad 0 < w < 1; \quad l = (1-z)^{-1} = \sum_{j=0}^{\infty} z^j; \quad 0 < z < 1;$$

The equality of leverage and the collateral ratio:

$$l = l/(l-1); \ \lim_{l \to 1} l/(l-1) = \infty ; \lim_{l \to \infty} l/(l-1) = 1$$

defines the debt fully collateralized by the capital. Its positive root, $l_2^* = 2$, explains why broker's loans extended to investors in the major stock exchanges do not exceed 50 percent of the buyer's portfolio[2].

Leverage and the Collateral Ratio



Financial flows and rates of return

The balance between aggregate financial flows:

$$d\,A(t) = d\,x(t) + d\,E(t)$$

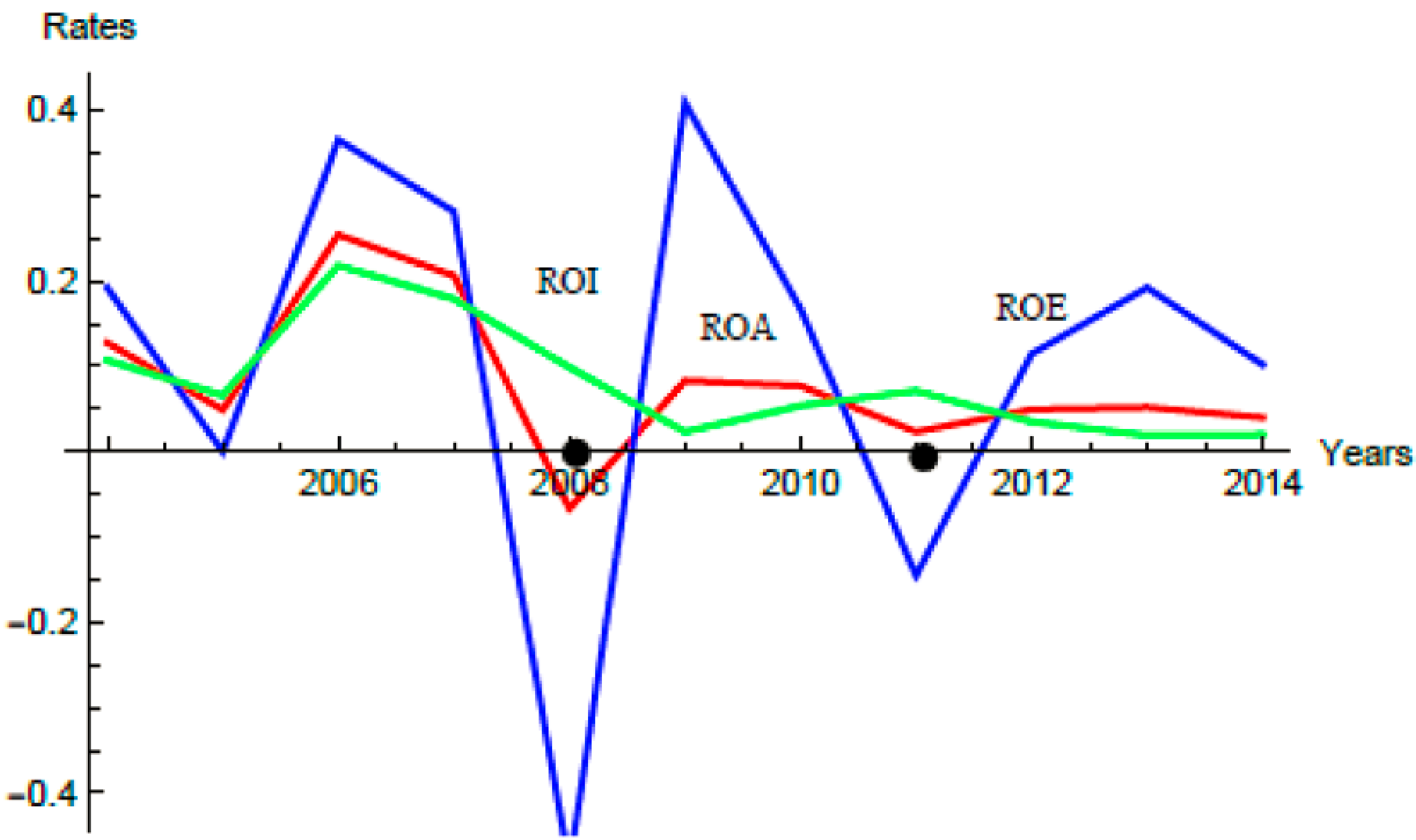
where d is the operator of taking differentials. Instantaneous rates of return:

on assets, $ROA \equiv \mu = dA(t)\,/\,A(t)dt$;

on aggregate debt, $ROR \equiv r = dx(t)\,/\,x(t)dt$;

on equity, $ROE \equiv \rho = dE(t)\,/\,E(t)dt$.

Global Rates of Return



The balanced financial market condition

The balance of financial flows:

$$\mu A = r\,x + \rho\,E$$

the macrofinancial *LtV* ratio, $z(t) = x(t) \,/\, A(t)$,
or the margin ratio, $w(t) = E(t) \,/\, A(t)$.

The balanced financial market (BFM) condition:

$$\mu = r z(t) + \rho(1 - z(t)) .$$

$$\mu = r + (\rho - r) w$$

Spreads and the short term debt demand and supply

The BFM condition is a static equality between indicators of (constant) demand for capital, $(\mu - r)$, and its supply, $(\rho - r)w$:

$$(\mu - r) = (\rho - r)w$$

.

The root is a relation of two spreads:

$$w^* = a / c \text{, or } \quad w^* = b / a$$

where $c = \rho - r$, and $b = a^2 / c$.

The ODE of the capital ratio dynamic

$$dw/dt = -aw(t) + a\ .$$

The economic reality is preserved for $(b/a) < 1$:

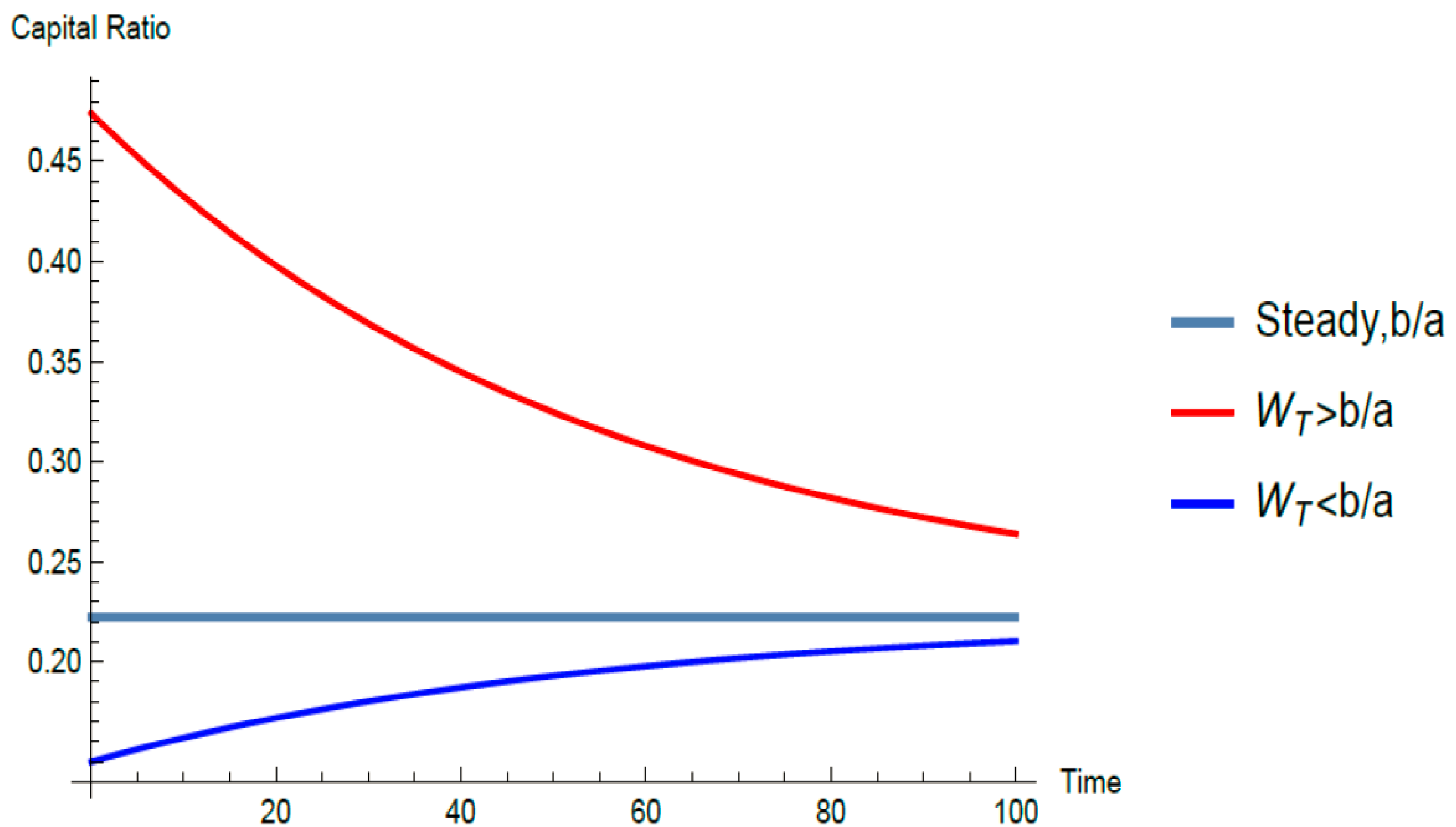
$$dw/dt = -aw(t) + b\ ; \quad 0 < w \leq 1\ .$$

where b measures the impact of real market factors.
The solution to ODE is

$$w(t) = Exp[-a\,t]\{w_T + b\int_0^t exp[a\,u]du\}$$

Trajectories for $a_{14} = 0.018;\ b_{14} = 0.004; b/a = 0.222$.

Deterministic Capital Ratio Process



The logistic equation for the macro-financial leverage:

$$dl(t) = [al(t) - bl^2(t)]dt\,; \quad 1 \le l < \infty\,.$$

where $a > 0$, and $b > 0$ measures feedbacks in the debt market.

The balanced financial market (BFM) condition:

$$\rho = r + (\mu - r)l \quad \text{for any } 1 \le l < \infty\,.$$

Given parameters r, μ, ρ, the root of (24):

$$K \equiv c / a = a / b$$

is a steady leverage for the logistic model.

The family of solutions to the logistic equation:

$$l(t) = K\{1 + (\frac{K}{l_0} - 1)\exp[-at]\}^{-1}$$

where l_0 is an initial leverage.

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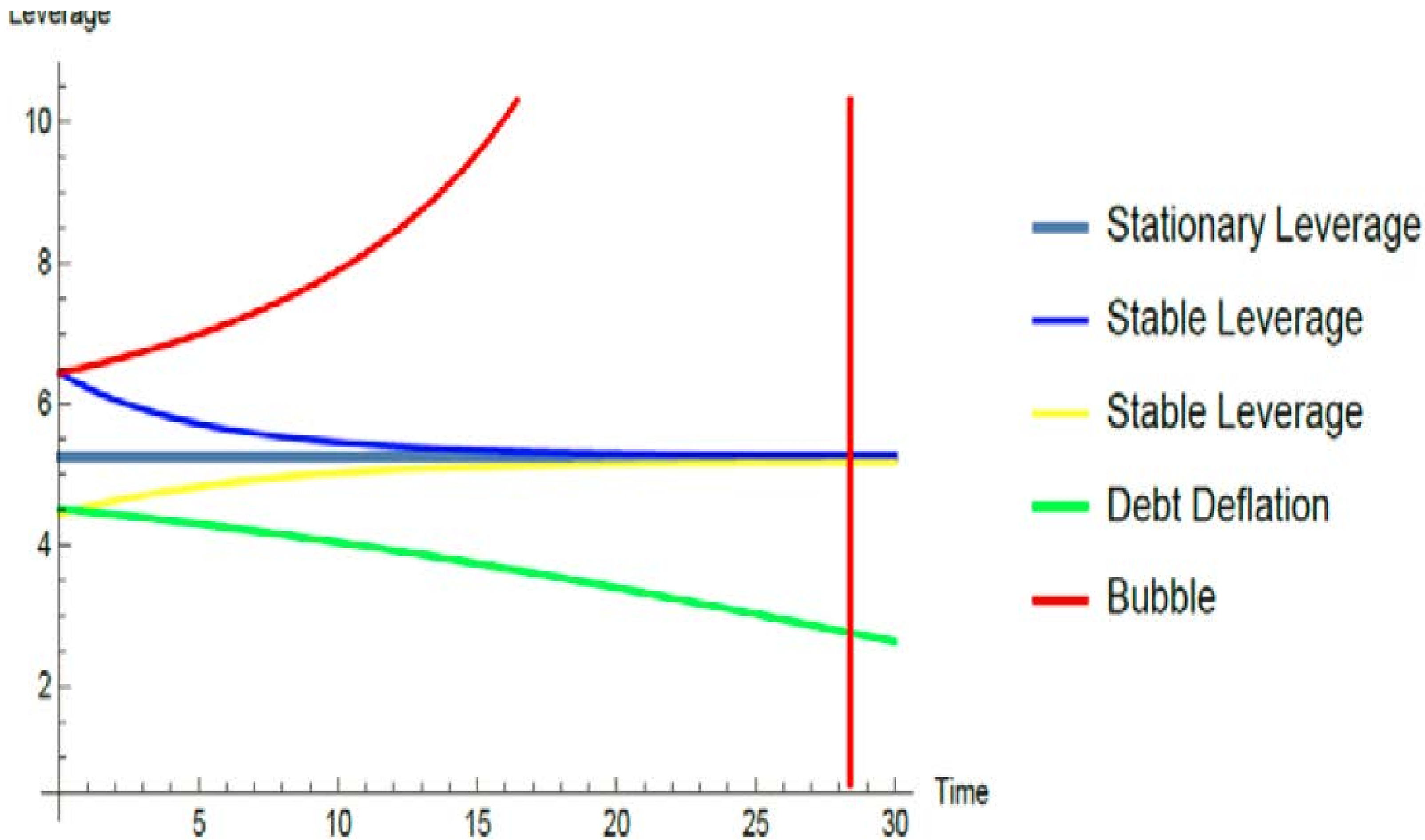


The coupled dynamics of the capital ratio and leverage

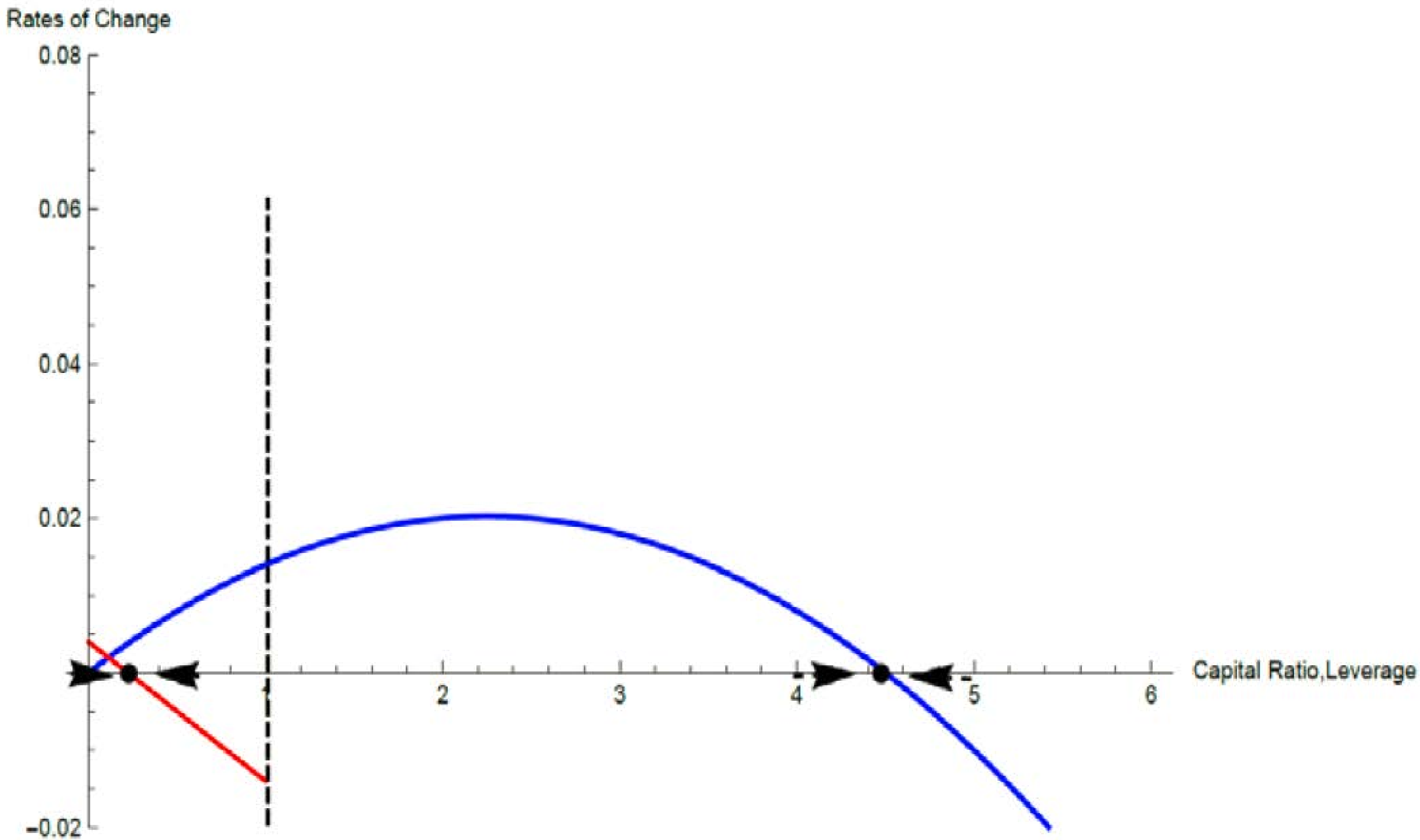
Macrofinancial interrelations:

$$dl / dw = -l^2; \ \varepsilon_{lw} = \left(w / l\right)\left(dl / dw\right) = -1$$

where $\varepsilon_{lw} = -1$ is the unitary elasticity.

The phase portrait with $w_{14} = 0.222; l_{14} = 4.5$.

Phase Portrait of the Coupled Capital and Leverage



The Ornstein-Uhlenbeck process for the capital ratio

The capital ratio dynamic under uncertainty:

$$d\,w(t) = [-a\,w(t) + b]dt + \sigma\, d\,B(t)$$

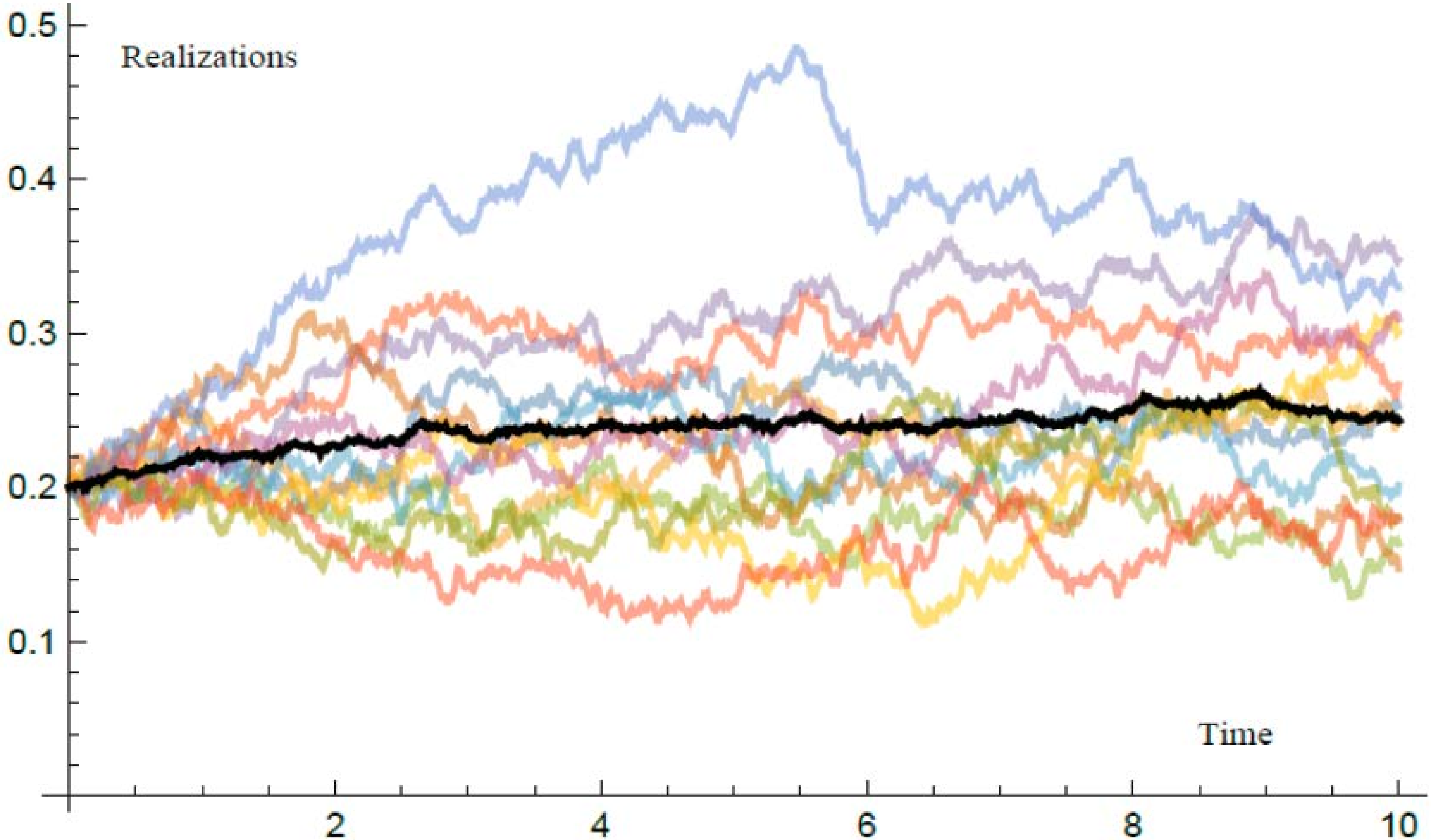
The he standard Brownian motion $B(t) = \int_0^t dB(u)$

is taken with a market volatility, σ.

The strong solution of the SDE:

$$w(t) = w_T \exp[-at] + \frac{b}{a}(1 - \exp[-at]) + \sigma \int_0^t \exp[a(z-t)]d\,B(t)$$

Stochastic Capital Ratio Process



The stationary PDF of the capital ratio

The forward Kolmogorov (or Fokker-Planck) equation:

$$\frac{\partial}{\partial t} p[w(t),t] = -\frac{\partial}{\partial w}\{[-aw(t)+b]p[w(t),t]\} + \frac{1}{2}\frac{\partial^2}{\partial w^2}\{\sigma^2 p[w(t),t]\}$$

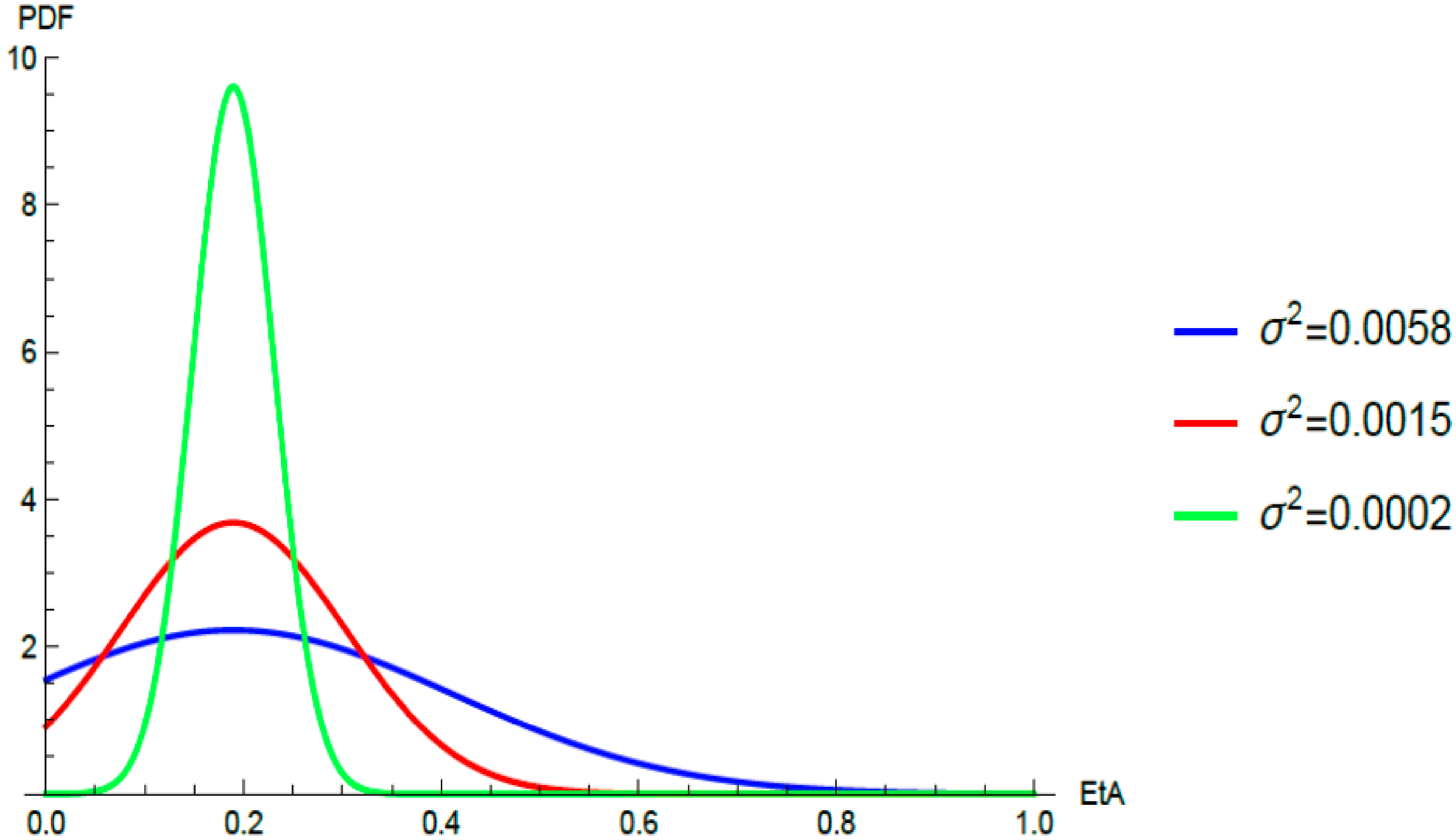
with some boundaries and initial conditions.

The stationary PDF of the capital ratio:

$$p(w) = \frac{N}{\sigma^2}\exp[-\frac{a}{\sigma^2}(w^2 - 2\frac{b}{a}w)]$$

where N is the constant of normalization.

Stationary PDF of the Capital Ratio



The logistic diffusion of leverage

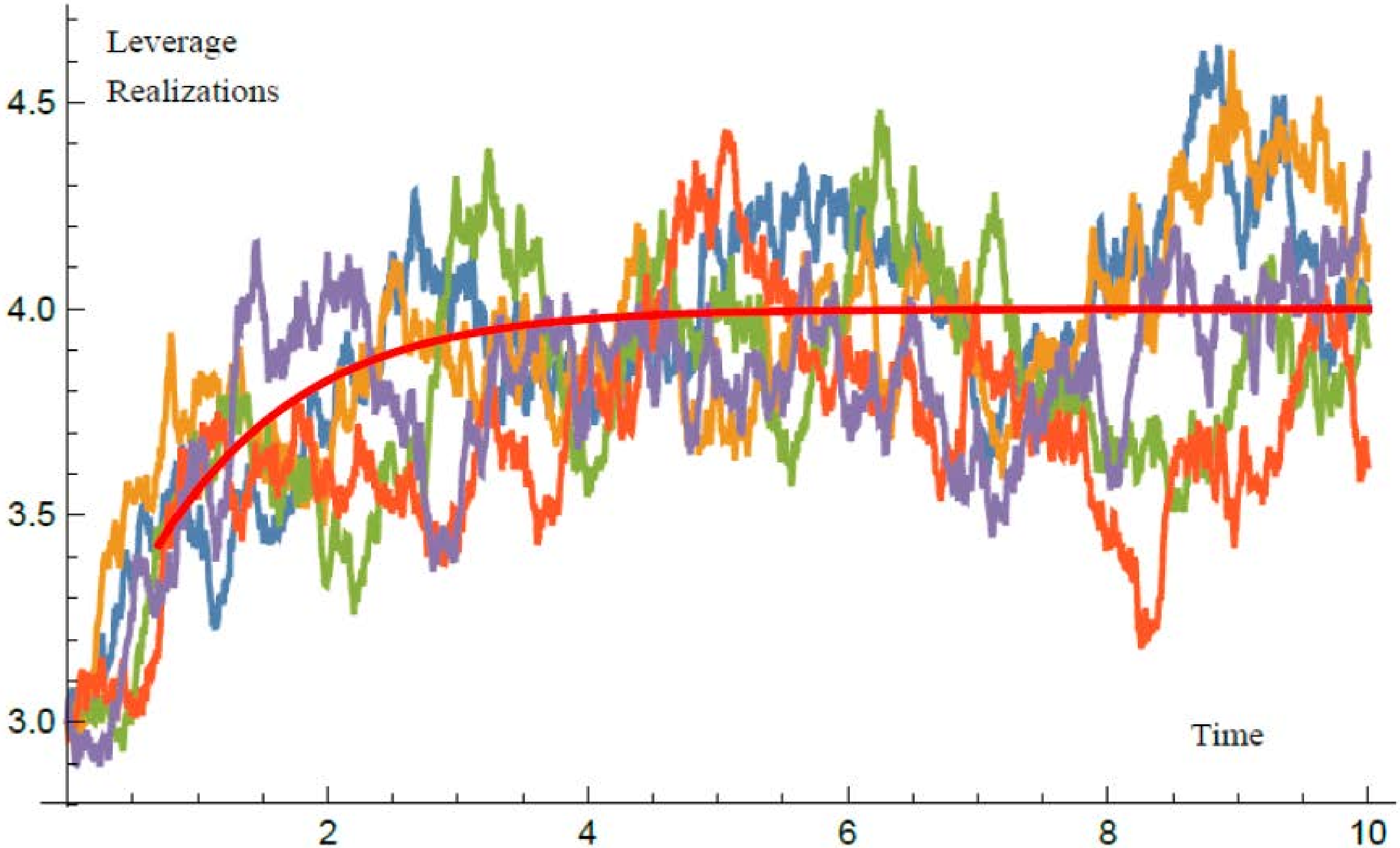
Continuous leverage process, $l(t)$, follows the logistic SDE:

$$dl(t) = [a - bl(t)]l(t)dt + \sigma l(t)dB(t).$$

Its strong solution:

$$l(t) = \frac{l_0 K \exp[(a - 0.5\sigma^2)t + \sigma W(t)]}{K + a l_0 \int_0^t \exp[(a - 0.5\sigma^2)u + \sigma W(u)]du};$$

Stochastic Leverage Process



Stationary gamma distribution of leverage

$$-\frac{\partial}{\partial l}[l(a-bl)p(l)]+\frac{1}{2}\frac{\partial^2}{\partial l^2}[\sigma^2 l^2 p(l)]=0$$

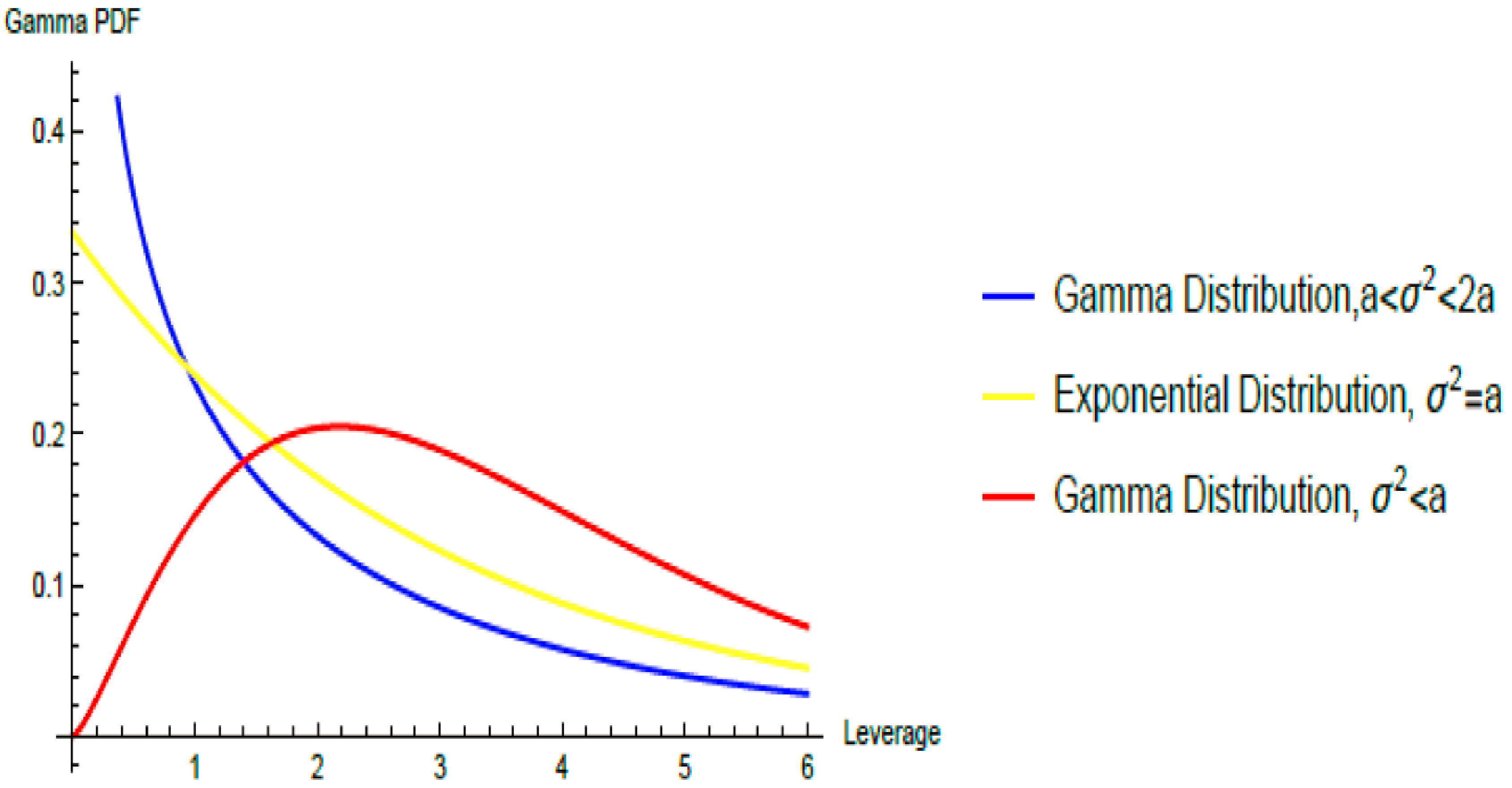
Stationary probability density function

$$p(l;\alpha,\beta)=[\beta^\alpha/\Gamma(\alpha)]l^{\alpha-1}e^{-\beta l}$$

defines gamma distribution with parameters of shape $\alpha=(2a/\sigma^2)-1$, and rate, $\beta=2b/\sigma^2$, or scale $1/\beta$.

For the positive parameter α (satisfying $0<\sigma^2<2a$) there exist three forms of gamma distribution.

Stationary Gamma Distributed Leverage



Characteristics of the gamma disribution

Expectation of the gamma distribution:

$$\langle L \rangle = \alpha/\beta = K - (\sigma^2 / 2b)$$

is larger than its mode:

$$Mode[l] = (\alpha - 1) / \beta = K - \sigma^2 / b;$$

the latter does not exist for a "J-shaped" distribution.

Convergence of leverage distributions to the stationary gamma distribution is of special importance.
The long term leverage convergence is measured by a stochastic Lyapunov exponent (SLE):

$$\lambda = \langle a - 2bL \rangle = a - 2b\langle L \rangle = \sigma^2 - a$$

where $\langle . \rangle$ means the ensemble averaging.

The long term debt demand and supply

Expected *ROE* depends upon leverage:

$$\rho(l) = r + (\mu - r)l \ .$$

It is an indicator of the long term debt supply because the rate of return on the borrowers' capital is higher, the larger loans they are able to take by leveraging their positions up, and vice versa.

Along the same reasoning the demand for debt:

$$\mu(l^{-1}) = r + (\rho - r)l^{-1} \ ,$$

whereby the knowledge of current rates ρ and r would help creditors to estimate their expected *ROA* on the aggregate portfolio.

The long term BFM condition

The inverse leverage, l^{-1}, should be measured in the same scale as leverage; otherwise, the separation of the debt demand and supply cannot be done effectively.

According to standard economic logic, the market price of assets might deviate, significantly sometimes, from its fair (true) value in the short run but persistent deviations are utterly ruinous. The true value of financial assets in aggregate exists, and might be estimated, as contingent to the equality between the long term debt demand and supply, the latter implying the real debt collateralization. Otherwise, the whole society could have been enriched by the unbounded borrowing, but *bona fide* it is similar to Baron Munchausen's ability to pull himself out of mire by his hair. In the absence of miracles,

The anchor leverage

Competitive adjustments ultimately equalize Returns on assets and equities:

$$\mu(l_N^{-1}) = \rho(l_N)\,.$$

The positive root, l_N, defines the *anchor leverage*:

$$l_N \equiv K^{0.5} = \left(a/b\right)^{0.5}$$

at which indicators of the debt supply and demand are equated.

The long term debt market configurations

The debt market configurations to the right of the anchor leverage, l_N, would drive leverage down until it comes into the "anchor" position:

$$[\rho(l) - \mu(l)] > 0 \Rightarrow dl / dt < 0 .$$

To the left of the anchor position leverage is smaller, the debt market is in disarray, and the debt demand indicator is higher than that of the debt supply:

$$[\rho(l) - \mu(l)] < 0 \Rightarrow dl / dt > 0 .$$

The natural rate of return (refinancing)

Resolving equations with regard to the rate of refinancing $r \equiv ROR$:

$$\mu(r) = \rho l^{-1} + (1 - l^{-1})r$$
$$\rho(r) = \mu l + (1 - l)r$$

explains the long term behaviour of savers and borrowers coordinated by their rates of return.

Condition : $\mu(r_N) = \rho(r_N)$, gives the "natural" rate of return

$$r_N = (\rho - \mu l^2) / (1 - l^2).$$

The natural rate of interest

Thus, the logistic leverage model provided some clarifications to a century-old discussion about the so-called "natural rate of interest". The great Swedish economist K. Wicksell (1899), who introduced the concept, argued the existence of an interest rate which, by equating saving and investment, facilitated stability of prices and minimal unemployment. Solution (43) has a narrower meaning of the "natural rate of return".

The anchor leverage and the mode of stationary gamma distribution

The natural assumption of the global financial system existence implies that its long run leverage has to be associated with the largest (elementary) probability of its realization. As it was shown, financial system exists if the aggregate debt supply equals to its demand which is tantamount to the equality of returns on assets, *ROA*, and on capital, *ROE*. This outcome is expected to take place at the anchor leverage, l_N, in other words, at the most probable value, or the mode, of the random gamma distributed leverage process.
The above said, formally, signifies the equality between the anchor leverage (36) and the mode of the stationary gamma distribution:

$$K^{0.5} = K - (\sigma^2 / b).$$

The critical variance and the random leverage convergence

The binding constraint is satisfied if gamma distributions $p[l(t),t]$ converge to the unimodal stationary gamma distribution which happens for the negative SLE, or for the expected variance no larger than its critical value:

$$\sigma_c^2 = a - \sqrt{ab} \, .$$

It sets the requirement for the implementation of coordinated, comprehensive and consistent reforms of financial markets: such reforms should decrease of the market uncertainty because only convergent random leverage process would stabilize debt markets in the long run.
Successful implementation of financial reforms, by keeping SLE negative, would stabilize leverage in the long run around its most probable value.

The PDF of leverage simulation

In the PDF simulation the most probable capital ratio was stuck at its deterministic steady state while the mode of gamma distributed leverage, if it existed, was associated with its "anchor" value at which aggregate debt supply and demand were balanced.
Thus, smaller than the steady state anchor leverage in the decoupled stochastic processes facilitated squeezing the bloated debt into a compact and efficient system of financial intermediation.

Deterministic skeletons in stationary KFP equations determine distributions rather than positions of a probabilistic system. Hence the stochastic model does not predict a causal chain of events; instead, particular elementary probabilities of continuous asymptotic capital ratio or leverage are explained by different market configurations associated with them.

Numerical simulation of leverage

The model simulation[3] was performed for structural parameters:

$a_{14} = 0.018$; $b_{14} = 0.004$; $a / b = 4.5$; variance of the capital ratio $\sigma_w^2 = 0.0014$, and variance of the "theoretical" leverage $\sigma_c^2 = 0.01$.

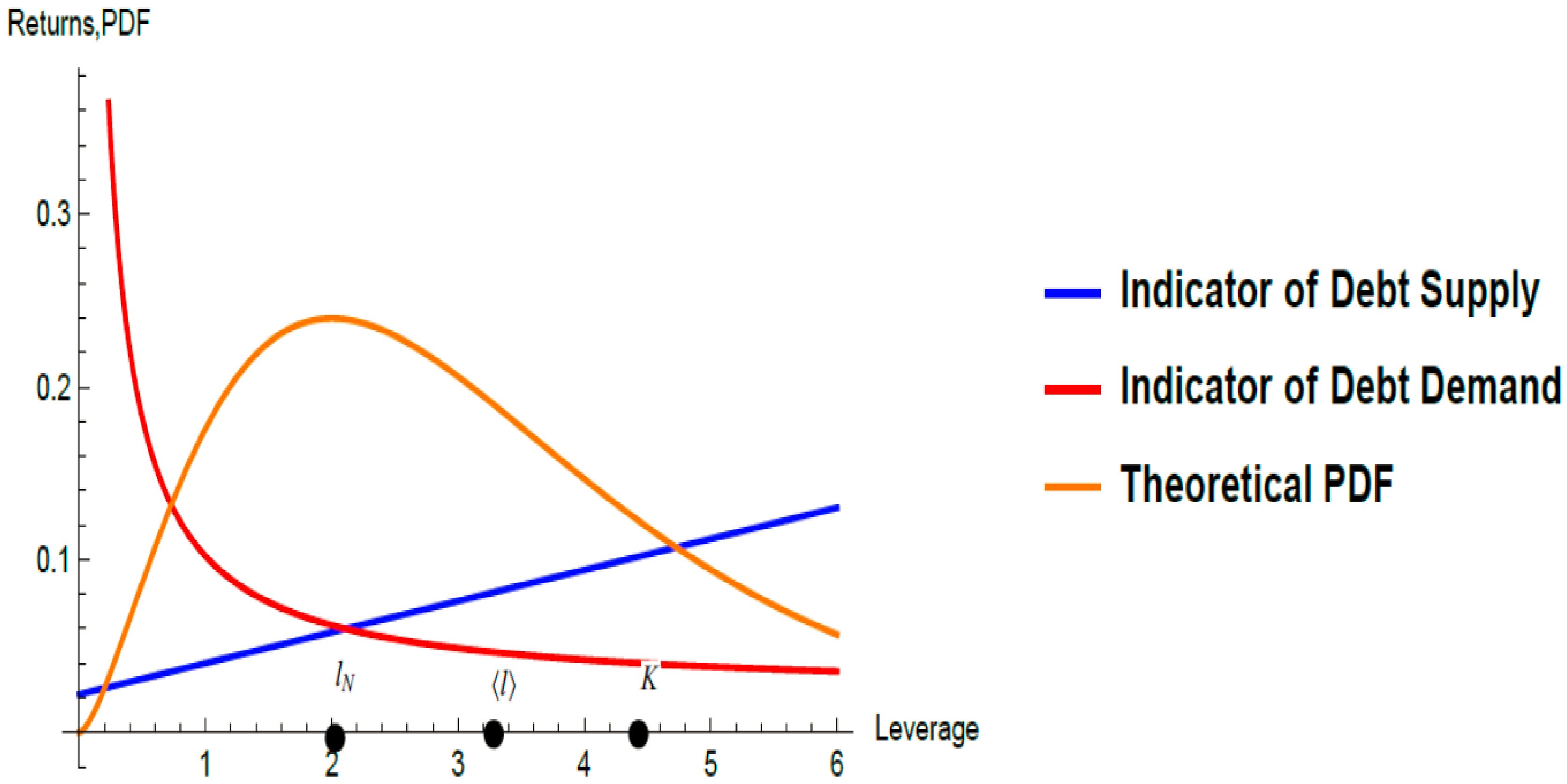
Simulated probability density functions:

$p(w) = 1.083\exp[-12.86w^2 + 5.71w]$ and $p(l) = 0.397\,l^{1.6}\exp[-0.8\,l]$.

Uni-Modal Gamma Distributed Leverage

Leverage	1	1.5	1.8	2.11	3.0	4.0	6.0
Spread, $(\rho - \mu) \times 10^4$	−6.2	−2.6	−1.2	0	2.7	5.2	9.5
PDF	0.178	0.229	0.24	0.243	0.21	0.149	0.057

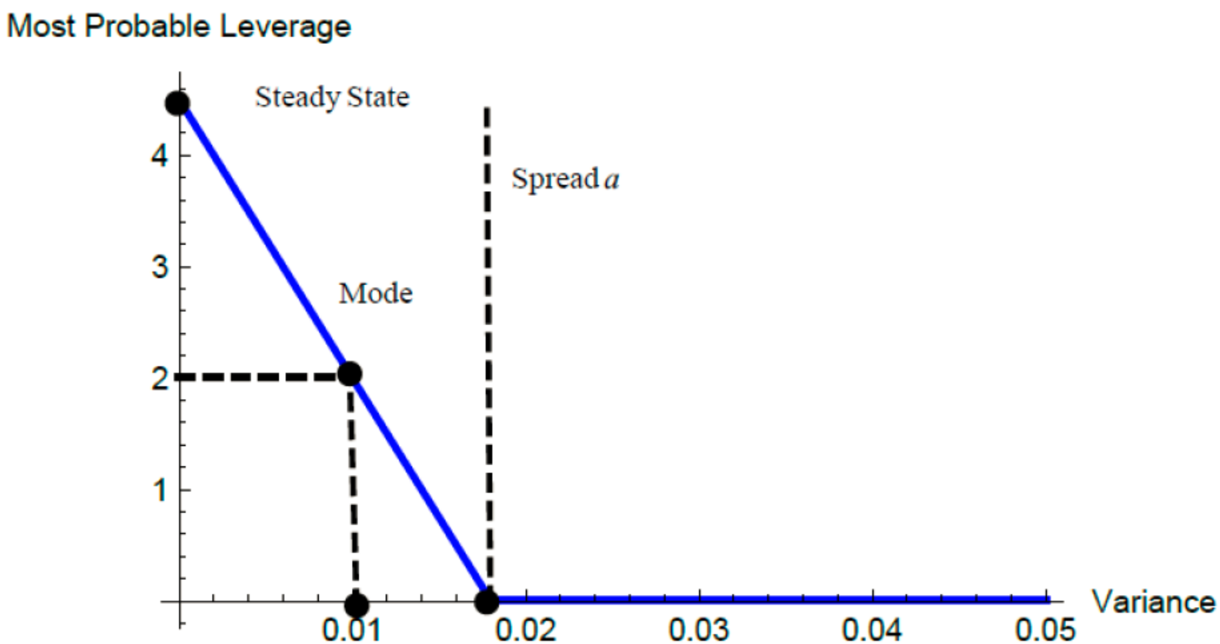
Scenario of the Global Leverage Convergence



The positive effects of the market noise

An interesting phenomenon of certain positive effects of a noisy (risky) debt market. The mode is equal to $l_N = a / b$ for $\sigma^2 = 0$, it decreases to the zero on the interval $0 < \sigma^2 < a$ and stays around zero in the interval $a < \sigma^2 < 2a$. Recall that its meaningful values lie in the interval $0 < \sigma^2 < (a - b)$.

Stationary Leverage and Global Risks



Scenarios of the Global Finance Taming

The proposed framework is a bit more than just "firing cannon at sparrows"; it is demonstrated by the estimates of annual amortization of global assets, debt and capital in the long run:

$$c_x = r(1-w^*)A \quad \text{and} \quad c_E = \rho w^* A .$$

Table 3. Annual amortization of global assets, debt and capital ($ trillion)

	Debt reimbursement, c_x	Capital amortization, c_E	Assets amortization, c_A
Annual in 2003-2014	11.63	3.44	15.06
Debt perpetuity simulation	4.95	-	-
Long run simulation, $w^* = 0.225; l_N = 2.12$	3.4	6.75	10.15
Long run simulation, $w_N = 0.474; l_N = 2.12$	3.4	14.21	17.61

The market noise factoring

Numerical results of the system simulation were received due to separation of stochastic debt and capital processes, and application of particular forms of the noise dependence upon the capital ratio or leverage.

In the SDE for capital ratio the noise factor was defined as $\sigma\, dB(t)$ because its deterministic skeleton had the lowest order component $b \equiv bw^0(t)$. Accordingly, since the lowest order leverage in the logistic SDE was $a\, l(t)$, its noise factor was added as $\sigma l(t) d\, B(t)$. Thus, the Ornstein-Ulenbeck and the logistic diffusion processes provided tractable and coherent scenarios of the random capital and debt dynamics in the long run.

The alternative noise hypotheses

Alternative processes:

$$dw(t) = [-aw(t) + b]dt + \sigma w(t) dB(t)$$

$$dl(t) = [a - bl(t)]\, l(t) dt + \sigma l^2(t) dB(t)$$

The appropriate stationary probability density functions:

$$p(w) = N w^{-\frac{2a}{\sigma^2} - 2} \exp[-\frac{2b}{\sigma^2}\frac{1}{w}]; \quad 0 \le w \le 1$$

$$p(l) = \frac{N}{\sigma^2} l^{-4} \exp[-\frac{1}{\sigma^2}(al^{-2} - 2bl^{-1})]; \quad l \ge 1 .$$

Comparison of the noise hypotheses

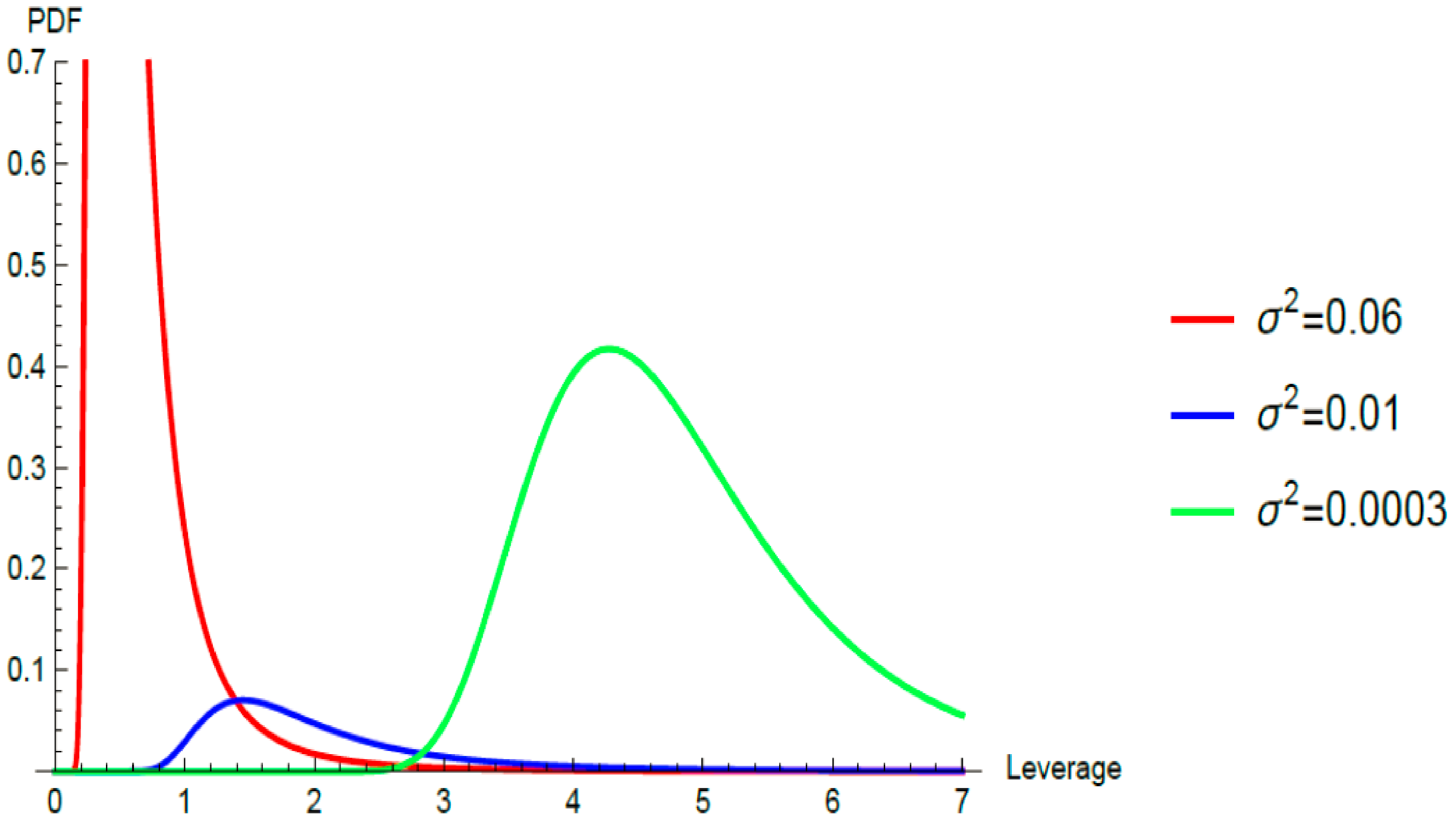
Note that, as was shown by S. Pascuali (2001) PDF existed for large variances, $\sigma^2 > 2a$, while for $\sigma^2 \to 0$ it tended to a degenerate Dirac distribution (located around $K = 4.4$ in our case).

Table 4. Location of different distributions

Variance	0.06	0.01	0.00001
LocationGamma	-	2.1	4.4
LocationAlt	0.37	1.45	4.5

The most probable leverage under the empirical variance (red curve) was located at $l^* = 0.37$, and it came into conflict with our basic economic presumptions since macrofinancial debt cannot be negative.

The Alternative Leverage Hypotheses



The model discussion

The alternative model supported the idea of Mao et al (2000) about the random process stabilization that could have been achieved by pumping noise into a system. Yet, in the context of our particular leverage model excessive noise drove the system to its virtual collapse thus realizing *bona fide* a “lethal stabilization”.

The gamma distribution does not exist in the interval $a < \sigma^2 < 2a$, and that posits some difficulties with its application; actually inconsistencies could be easily avoided via averaging empirical data over some periods.
The alternative distribution exists without any constraints imposed upon the variance, but its most probable value is meaningless.
So can it be qualified as a substantive improvement of the basic framework, from the economic point of view?