Auctions with Limited Commitment*

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Abstract

We study the role of limited commitment in a standard auction environment. In each period, the seller can commit to an auction with a reserve price but not future auctions. We characterize the set of equilibrium profits attainable for the seller as her commitment power vanishes. If the number of buyers exceeds a distribution-specific cutoff, an efficient auction is the unique limit of equilibrium outcomes; otherwise, profits above the efficient auction profit are achievable. We give exact conditions under which the maximal profit is attained through an initial auction with a reserve price, followed by a continuously decreasing price path.

1 Introduction

It is well understood that in standard auctions such as first-price or second-price auctions, the seller can increase her profit by imposing a minimal bid (or reserve price) (Myerson, 1981; Riley and Samuelson, 1981). A reserve price leads to inefficient exclusion of low-valued

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buyers which helps extract higher payments from high-valued buyers. If no bidder bids above the reserve price, the seller has to commit to not auctioning the object again, even though there is common knowledge of unrealized gains from trade with the excluded buyers.

This aspect of full commitment is not entirely satisfactory in many applications. In practice, buyers may not find it credible that the seller can withhold an unsold object from the market forever, as aborted auctions are common and unsold objects are frequently re-auctioned or offered for sale later. Economists have long recognized the importance of understanding the role of commitment in the auction setting. Milgrom (1987) and McAfee and Vincent (1997) consider a seller with limited commitment in the sense that in each period she can commit to an auction with a reserve price but not future auctions. They show that efficient auctions are the only stationary equilibrium outcome because limited commitment will force the reserve price to drop to her reservation value.[1] However, their analysis and stark result crucially rely on stationarity—which is known to severely restrict equilibrium behavior and payoffs—leaving open the question of the exact equilibrium implication of limited commitment. A better understanding of the role of limited commitment in auctions not only is economically relevant but also enriches the existing auction theory.

We adopt the modeling approach of Milgrom-McAfee-Vincent to limited commitment. There are one seller with a single indivisible object and multiple buyers whose values are drawn independently from a common distribution. In each period until the object is sold, the seller posts a reserve price and holds an auction. For simplicity, we restrict the exposition to second-price auctions, but our results do not change if the seller can choose from a larger class of auctions in each period. Each buyer can either wait for the next auction, or submit a bid no smaller than the reserve price. Waiting is costly and both the buyers and the seller discount at the same rate. Within a period, the seller commits to the rules of the auction and the announced reserve price. The seller cannot, however, commit to future reserve prices.

This framework is sufficiently rich to investigate the role of commitment. The seller's commitment power varies with the period length (or effectively with the discount factor). If the period length is infinite, the seller has full commitment power. As the period length shrinks, the seller's commitment power diminishes. We adopt the solution concept of perfect Bayesian equilibrium, which is well-defined for the discrete-time game, and restrict attention to buyer-symmetric equilibria. Within the framework, we analyze the continuous-time limit at which the seller's commitment power vanishes. We ask the following questions: what is the set of equilibrium payoffs that is attainable by the seller? What is the equilibrium selling

[1] Milgrom constructs a buyer-symmetric, stationary equilibrium directly in continuous time, while McAfee and Vincent focus on a discrete-time model.

strategy that attains the maximal payoff? When can the seller credibly use reserve prices above her reservation value to increase her profit?

We obtain the following results. First, the full commitment profit cannot be achieved under limited commitment. In order to attain the full commitment profit, the seller would have to maintain a constant reserve price above her reservation value (Myerson, 1981). This is not sequentially rational. Once the initial auction fails, the seller can deviate and end the game with a positive profit by running an efficient auction—that is, by setting a reserve price equal to her reservation value. Second, if the number of bidders exceeds a distribution-specific cutoff, an efficient auction maximizes the seller's profit and implements the unique limit of the equilibrium outcomes. For many widely used distributions, the temptation to gather the profit immediately through a zero reserve price from a small number of buyers is sufficient to induce the seller to give up screening completely. For instance, if the type distribution has a finite density, then an efficient auction is revenue-maximizing if there are more than two buyers. Third, if the number of bidders falls short of the aforementioned cutoff, strictly positive reserve prices can arise in equilibrium and the efficient auction is not optimal. Finally, under the assumption that the monopoly profit function is concave, we obtain an ordinary differential equation that describes the optimal limit outcome if the efficient auction is not optimal. We characterize the exact maximal revenue and show that it can be attained through an initial auction with a strictly positive reserve price followed by a sequence of continuously declining reserve prices.

The modeling framework of Milgrom (1987) and McAfee and Vincent (1997) allows us to focus on the characterization of the equilibrium set of the seller's payoff when she cannot exclude buyers forever and is restricted to use standard auctions with reserve prices, which are popular in practice. A related and complementary question is: what if the seller can use more general mechanisms? We conjecture that in our framework, the restriction to standard auctions with reserve prices is without loss in characterizing the seller's maximal payoff. This has been shown by Skreta (2016) in a finite horizon model. The intuition for this conjecture can be gained from the literature on the ratchet effect. It is costly for the seller to elicit information without committing how to use it. This would expose a buyer to exploitation by the seller if he reveals private information, and the seller must compensate the buyer for that. This intuition suggests that it is suboptimal to use mechanisms where a bid (more generally a message) does not lead to an immediate allocation. If every bid indeed leads to an immediate allocation, it remains to show if and how the seller can screen buyers over time, which is the focus of our paper.

It is worthwhile to compare our results to those in Milgrom (1987) and McAfee and Vincent (1997) whose analysis relies on the assumption of stationarity. The logic behind our results is entirely different. We show that the optimality of the efficient auction has nothing to do with stationarity, and we give precise conditions under which the efficient auction is and is not revenue-maximizing without stationarity assumption. This helps clarify the role of limited commitment in the auction setting. Our results are also related to Bulow and Klemperer (1996) who show that running the efficient auction with one more bidder, though not optimal, is more beneficial to the seller with full commitment than setting the reserve price optimally. We show that if the number of bidders is not too small, the efficient auction is in fact revenue-maximizing for a seller with limited commitment.

Since the full commitment profit is not attainable, we first have to identify the maximal attainable profit. Moreover, with an infinite horizon we cannot use backward induction to identify equilibria, and with limited commitment we cannot rely on the revelation principle.[2] Therefore, we need a new method to characterize the set of the seller's equilibrium profits.

The key idea we employ is to translate the limited-commitment problem into an auxiliary mechanism design problem with full commitment, but with a crucial extra constraint. The extra constraint is intended to capture limited commitment. In the original limited-commitment problem, at any stage of the game, the seller can always run an efficient auction to end the game, so her continuation value in any equilibrium must be bounded below by the payoff from an efficient auction for the corresponding posterior belief.[3] This equilibrium payoff restriction, which we refer to as the "payoff floor" constraint, is a necessary implication of the seller's sequential rationality. Hence, the value of the auxiliary problem provides an upper bound for the equilibrium payoffs in the original game (in the continuous-time limit). We proceed to solve the auxiliary problem and show that its value and its solution can be approximated by a sequence of equilibrium outcomes of the original game. Therefore, the value of the auxiliary problem is precisely the maximal attainable equilibrium payoff in our original problem, and the solution to the auxiliary problem is precisely the limiting selling strategy that attains this maximal payoff.

As in Milgrom (1987) and McAfee and Vincent (1997), there is no definitive last period in our model when the seller can fully commit. Milgrom (1987) restricts attention to stationary

[2]Bester and Strausz (2001) develop a version of the revelation principle with limited commitment for environments with one agent and a finite number of periods. It does not apply to our setting because our model has multiple buyers (Bester and Strausz, 2000).

[3]For a given auxiliary mechanism, the seller knows exactly which set of types are left at each moment in time, if the mechanism is carried out. Consequently, she can compute the posterior beliefs as well as her continuation payoff from the given mechanism.

equilibria, while McAfee and Vincent (1997) assume that the seller's valuation is strictly below the support of the buyers' valuations, which leads to an endogenous finite horizon, allowing them to use backward induction to characterize equilibria. A general mechanism design framework with a finite horizon is developed by Skreta (2006, 2016) who shows by backward induction that the optimal mechanism is a sequence of standard auctions with reserve prices.[4] In contrast, we restrict attention to auction mechanisms in each period and characterize the full set of equilibrium profits as the commitment power vanishes.

An alternative approach to modeling limited commitment is to assume that the seller cannot commit to trading rules even for the present period. McAdams and Schwarz (2007) consider an extensive form game in which the seller can solicit multiple rounds of offers from buyers. Their paper shows that if the cost of soliciting another round of offers is large, the seller can credibly commit to a first-price auction, and if the cost is small, the equilibrium outcome approximates that of an English auction. In Vartiainen (2013), a mechanism is a pure communication device that permits the seller to receive messages from bidders. The seller cannot commit to any action after receiving the messages, and there is no discounting. Vartiainen shows that the only credible mechanism is an English auction. In contrast to these papers, we posit that the seller cannot renege on the agreed terms of the trade in the current period. For example, this might be enforced by the legal environment.

A special case of our setup is the model of bilateral bargaining or durable goods monopoly, in which an uninformed seller makes price offers to a single privately informed buyer, as already recognized by Milgrom (1987) and McAfee and Vincent (1997). In his seminal paper, Coase (1972) argues that a price-setting monopolist completely loses her monopoly power and prices drop quickly to her marginal cost if she can revise prices frequently. Fudenberg, Levine, and Tirole (1985) and Gul, Sonnenschein, and Wilson (1986) confirm that stationary equilibria satisfy the Coase conjecture. If the seller's reservation value lies in the support of the single buyer's valuation, however, Ausubel and Deneckere (1989) show that in addition to the stationary Coasian equilibria, there is a continuum of non-stationary "reputational equilibria". In these equilibria, the price sequence posted by the seller may start with some arbitrary price which decreases over time, and any seller deviation from the equilibrium price path is deterred by the threat to switch to the low-profit Coasian equilibrium path. In the limit as the period length diminishes, these trigger-strategy equilibria allow the seller to

[4] Hörner and Samuelson (2011) and Chen (2012) analyze the dynamics of posted prices under limited commitment in a finite horizon model. They assume that the winner is selected randomly when multiple buyers accept the posted price.

achieve any profit between zero and the monopoly profit.[5]

Different from Ausubel and Deneckere (1989), our model has multiple buyers. A natural idea is to replicate their trigger strategy equilibrium with efficient auctions as off-path punishment. However, this type of reputation equilibrium does not always exist (e.g. when there are three bidders with uniform distribution) and when it does, the equilibrium profit is still strictly lower than the full commitment profit. It opens the question of whether strategies more complicated than the simple trigger strategy can yield a higher profit. Therefore, an extension of Coase conjecture argument and the equilibrium construction of Ausubel and Deneckere (1989) is not useful for us to characterize the maximal profit. We propose the auxiliary mechanism design problem with full commitment to characterize the maximal profit attainable in all equilibria of the original problem with limited commitment, and show that the simple trigger strategy attains the maximal profit.

The paper is organized as follows. In the next section, we formally introduce the model. Section 3 uses a uniform example to illustrate how to construct a particular class of equilibria and why equilibrium construction alone does not yield our main results. Section 4 states the formal results. Section 5 presents our methodological approach. In Section 6 we comment on alternative modeling assumptions. Unless noted otherwise, proofs can be found in Appendix A. Omitted proofs can be found in the Supplemental Material.

2 Model

We consider the standard auction environment where a seller (she) wants to sell an indivisible object to n potential buyers (he). Buyer i privately observes his own valuation for the object $v^i \in [0,1]$. We use $(v^i, v^{-i}) \in [0,1]^n$ to denote the vector of the n buyers' valuations, and $v \in [0,1]$ to denote a generic buyer's valuation. Each v^i is drawn independently from a common distribution with full support, c.d.f. $F(\cdot)$, and a continuously differentiable density $f(\cdot)$ such that $f(v) > 0$ for all $v \in (0,1)$. The highest order statistic of the n valuations (v^i, v^{-i}) is denoted by $v^{(n)}$, its c.d.f. by $F^{(n)}$, and the density by $f^{(n)}$. The seller's reservation value for the object is constant over time and we normalize it to zero.[6]

[5] Wolitzky (2010) analyzes a Coasian bargaining model in which the seller cannot commit to delivery. In his model, the full commitment profit is achievable even in discrete time because there is always a no-trade equilibrium which yields zero profit.

[6] The reservation value can be interpreted as a production cost. Alternatively, if the seller has a constant flow value of using the object, the opportunity cost is the net present value of the seller's stream of flow values. What is important here is that the seller's reservation value is the same as the value of the lowest possible buyer type. In Section 6, we discuss the case that the seller's reservation value is in the interior of the type distribution which introduces uncertainty about the number of potential buyers.

Time is discrete and the period length is denoted by Δ. In each period $t = 0, \Delta, 2\Delta, \ldots,$ the seller runs a second-price auction (SPA) with a reserve price. To simplify notation, we often do not explicitly specify the dependence of the game on Δ. The timing within period t is as follows. First, the seller publicly announces a reserve price p_t for the auction run in period t, and invites all buyers to submit a valid bid, which is restricted to the interval $[p_t, 1]$. After observing p_t, all buyers decide simultaneously either to bid or to wait. If at least one valid bid is submitted, the winner and the payment are determined according to the rules of the second-price auction and the game ends. If no valid bid is submitted, the game proceeds to the next period. Both the seller and the buyers are risk-neutral and have a common discount rate $r > 0$. This implies a discount factor per period equal to $\delta = e^{-r\Delta} < 1$. If buyer i wins in period t and has to make a payment π^i, then his payoff is $e^{-rt}(v^i - \pi^i)$, and the seller's payoff is $e^{-rt}\pi^i$.

We assume that the seller has limited commitment power. She can commit to the reserve price that she announces for the current period: if a valid bid is placed, then the object is sold according to the rules of the announced auction and she cannot renege. She cannot commit, however, to future reserve prices: if the object was not sold in a period, the seller can always run another auction with a new reserve price in the next period. She cannot promise to stop auctioning an unsold object, or commit to a predetermined sequence of reserve prices.

We denote by $h_t = (p_0, p_\Delta, \ldots, p_{t-\Delta})$ the public history at the beginning of $t > 0$ if no bidder has placed a valid bid up to t, and write $h_0 = \emptyset$ for the history at which the seller chooses the first reserve price.[7] Let H_t be the set of such histories. A (behavior) strategy for the seller specifies a Borel-measurable function $p_t : H_t \to P[0,1]$ for each $t = 0, \Delta, 2\Delta, \ldots,$ where $P[0,1]$ is the space of Borel probability measures endowed with the weak* topology.[8] A (behavior) strategy for buyer i specifies a function $b_t^i : H_t \times [0,1] \times [0,1] \to P[0,1]$ for each $t = 0, \Delta, 2\Delta, \ldots,$ where we assume that $b_t^i(h_t, p_t, v^i)$ is Borel-measurable in v^i, for all $h_t \in H_t$, and all $p_t \in [0,1]$, and that supp $b_t^i(h_t, p_t, v^i) \subset \{0\} \cup [p_t, 1]$, where "0" denotes no bid or an invalid bid.

We consider perfect Bayesian equilibria (PBE), and we will focus on equilibria that are buyer symmetric.[9] We will not distinguish between strategies that coincide with probability one for all histories. In the rest of the paper, "equilibrium" is used to refer to this class of symmetric perfect Bayesian equilibria. Let $\mathcal{E}(\Delta)$ denote the set of equilibria of the game for

[7] We do not have to consider other histories because the game ends if someone places a valid bid.

[8] We slightly abuse notation by using p_t both for the seller's strategy and the announced reserve price at a given history.

[9] See Fudenberg and Tirole (1991) for the definition of PBE in finite games. The extension to infinite games is straightforward.

given Δ.[10] Let $\Pi^{\Delta}(p,b)$ denote seller's expected revenue in any equilibrium $(p,b) \in \mathcal{E}(\Delta)$. We are interested in the entire set of profits that the seller can achieve in the limit when the period length vanishes. The maximal profit in the limit is

$$\Pi^* := \limsup_{\Delta \to 0} \sup_{(p,b) \in \mathcal{E}(\Delta)} \Pi^{\Delta}(p,b).$$

The minimal profit in the limit is

$$\Pi_* := \liminf_{\Delta \to 0} \inf_{(p,b) \in \mathcal{E}(\Delta)} \Pi^{\Delta}(p,b).$$

The analysis of the continuous-time limit allows us to formulate a tractable optimization problem. We will justify our approach by providing approximations through discrete time equilibria. An alternative approach is to set up the model directly in continuous time. This approach, however, has unresolved conceptual issues regarding the definition of strategies and equilibrium concepts in continuous-time games of perfect monitoring, which are beyond the scope of this paper.[11]

Remark 1 (**Larger Class of Permissible Auction Formats**)**.** Our exposition and analysis are formulated in terms of second-price auctions. In Appendix E, we establish a revenue equivalence result for our problem, and show that all of our results hold for a larger class of symmetric bidding mechanisms in which only the winner pays. This class includes not only standard first price and second price auctions with reserve prices, but also exotic mechanisms like third price auctions and auctions where the winner's payment may depend on his own bid and his rivals' bids. In these mechanisms, the object is always allocated to the bidder with the highest valid bid. The main substantial restriction is allocative efficiency. This rules out posted prices with a rationing rule (as for example in Hörner and Samuelson, 2011), lotteries, or raffles. Formally, we show that any equilibrium allocation and equilibrium payoff in the game where the seller can choose a (potentially different) mechanism from this larger class of mechanisms in every period can be replicated in the game where the seller is restricted to choose only second-price auctions with reserve prices and vice versa.

Remark 2 (**Interpretation of the Continuous Time Limit**)**.** We take $\Delta \to 0$ in computing the limiting payoff. This need not be interpreted literally as running auctions frequently in real time. As in the dynamic games literature, this formulation is equivalent to taking

[10] We establish equilibrium existence in Proposition 2.(i) (see Appendix A.1).

[11] See Bergin and MacLeod (1993) and Fuchs and Skrzypacz (2010) for related discussions.

$\delta \to 1$ in a discrete-time problem. The continuous-time limit, however, is more convenient when we consider limiting price paths.

Remark 3 (Relation to Milgrom, McAfee-Vincent). In the terminology of the durable goods monopoly literature, we consider the "no-gap" case as in Milgrom (1987). It is well-known that in the "gap case", where F has a support $[\varepsilon, 1]$ for $\varepsilon > 0$, the game essentially has a finite horizon and all equilibria are stationary (i.e., buyers' strategies depend only on their own valuation and the current reserve price), which is the focus of McAfee and Vincent (1997).

Before we proceed, we introduce several mild assumptions on the distribution function F. Most of our analysis only depends on a subset of these assumptions, and we will note explicitly which assumption is used for which result.[12] Examples of distributions that simultaneously satisfy all assumptions include the uniform distribution and more generally all power function distributions $F(v) = v^k$ with support $[0, 1]$ and $k > 0$. Let the virtual valuation be denoted by $J(v) := v - (1 - F(v))/f(v)$.

Assumption 1. *There exists* $\underline{v} \in (0, 1)$ *such that* $J(v) > 0$ *and strictly increasing on* $(\underline{v}, 1]$*, and* $J(v) \leq 0$ *on* $[0, \underline{v}]$.

Assumption 1 is lightly weaker than the standard monotone virtual value assumption widely adopted in the mechanism design literature. For $v > \underline{v}$, it corresponds to assuming decreasing marginal revenues (see Bulow and Roberts, 1989). As will be clear later, this assumption allows us to neatly connect our model of limited commitment with Myerson's optimal auction with full commitment. See Section 6 for detailed discussion of the role of this assumption for our analysis. The following two assumptions are regularity conditions on the distribution in the neighborhood of 0.

Assumption 2. $\phi := \lim_{v\to 0} (f'(v)v)/f(v)$ *exists and* $\phi < \infty$.

It is easy to see that $\phi = \lim_{v\to 0} (f(v)v)/F(v) - 1$, so $\phi \geq -1$ if the limit exists. Assumption 2 rules out the knife-edge cases of $\phi = -1$ and $\phi = \infty$.[13] Assumption 2 is satisfied, for example, if the density function f is bounded away from 0 and has a bounded derivative. It is also satisfied for a class of distributions which includes densities with $f(0) = 0$ or $f(0) = \infty$ such as the power function distributions $F(v) = v^k$ with $k > 0$.

[12] All four assumptions are independent. Details can be found in Appendix F in the Supplemental Material.

[13] An example for the knife-edge cases, due to Yuliy Sannikov, is the distribution function $F(v) = v^{(\ln(1/v))^k}$ defined on $[0, 1]$. For this distribution function, $\phi = -1$ if $k = -1/2$, and $\phi = \infty$ if $k = 1/2$.

Assumption 3. *There exist constants* $0 < M \leq 1 \leq L < \infty$ *and* $\alpha > 0$ *such that* $Mv^\alpha \leq F(v) \leq Lv^\alpha$ *for all* $v \in [0, 1]$.

Assumption 3 is adopted from Ausubel and Deneckere (1989) who use it to prove the uniform Coase conjecture. We use it when we extend this result to the auction setting.

Assumption 4. *The revenue function* $v(1 - F(v))$ *is concave on* $[0, 1]$.

Assumption 4 is equivalent to assuming that $J(v)f(v)$ is increasing. It is also equivalent to $(f'(v)v)/f(v) > -2$. Note that, under Assumptions 1 and 2, $\phi = \lim_{v \to 0} (f'(v)v)/f(v) > -1$, so $v(1 - F(v))$ is concave for v sufficiently close to 0. This will allow us to dispense with Assumption 4 for all but one of our results.

3 A Heuristic Example

To get an intuitive idea of non-stationary equilibria, let us heuristically construct a particular class of equilibria in continuous time in a simple example. This example will also illustrate why the constructive method is not useful for us to characterize the maximal profit. Consider two bidders ($n = 2$) whose values are uniformly distributed on $[0, 1]$. At any $t \geq 0$, on the equilibrium path, the seller posts a reserve price p_t. Buyers use a cutoff strategy, that is, a buyer bids before time t if and only if his value v is weakly above some cutoff v_t, so that v_t is the highest type remaining at time t. If the seller deviates from the reserve price path p_t, the off-path play stipulates that the seller posts a constant reserve price $p_t \equiv 0$ and buyers place valid bids if and only if $p_t = 0$.[14]

The Buyers' Incentives Consider a buyer whose valuation equals the cutoff type v_t at $t > 0$. This buyer must be indifferent between buying at p_t, and waiting for a period of length dt to accept a lower price p_{t+dt}. The latter exposes him to the risk of losing, if his opponent has a valuation between v_{t+dt} and v_t. Therefore, the indifference condition is

$$v_t - p_t = (1 - rdt) \left(\frac{v_{t+dt}}{v_t} \right) (v_t - p_{t+dt}) \,. \tag{3.1}$$

The left-hand side of equation (3.1) is the marginal bidder's profit from trading immediately at t, conditional on being the bidder with the higher valuation. The right-hand side is the

[14] We can ignore continuations after deviations by a buyer because they either remain undetected or lead to a successful sale which ends the game. For this heuristic construction, we also assume that both p_t and v_t are continuously differentiable and decreasing over time.

option value from waiting: $(1-rdt)$ is the discounting, $\frac{v_{t+dt}}{v_t}$ is the probability that the opponent's valuation is below v_{t+dt} conditional on the fact that her valuation is below v_t (this is the probability that v_t wins the object at $t+dt$), and $v_t - p_{t+dt}$ is the payoff the marginal bidder gets from the delayed trade at $t+dt$. Using a first-order approximation, we obtain the following differential equation governing p_t and v_t:

$$\dot{p}_t = \left(\frac{\dot{v}_t}{v_t} - r\right)(v_t - p_t). \tag{3.2}$$

The Seller's Incentive We look for an equilibrium in which the seller is indifferent between following the equilibrium path and deviating at any time $t > 0$. This condition is given by,

$$\int_t^\infty e^{-r(s-t)} p_s \frac{2v_s}{(v_t)^2}(-\dot{v}_s)\, ds = \frac{1}{3} v_t. \tag{3.3}$$

The left-hand side is the expected present value of the seller's equilibrium revenue at $t > 0$: Since v_t is continuously differentiable, at each moment $s > t$, only the marginal buyer type v_s buys at the reserve price p_s. The marginal type has a conditional density $2v_s/(v_t)^2$, the density of the higher value of two buyers, and it declines with the rate $-\dot{v}_s$. The right-hand side is the seller's revenue after a deviation: running an efficient second-price auction with an expected revenue of $\Pi^E(v_t) = \frac{1}{3}v_t$.

Combining the Seller's and the Buyers' Incentives Equations (3.2) and (3.3) together give rise to a second-order differential equation in v_t:[15]

$$\ddot{v}_t + r\dot{v}_t = 0. \tag{3.4}$$

Boundary conditions for this ODE are given by the cutoff after the first instant which we write as v_0^+, and the fact that the seller cannot maintain a positive price forever, which implies $\lim_{t\to\infty} v_t = 0$. Using these boundary conditions, we obtain the following solution for the cutoff path

$$v_t = v_0^+ e^{-rt}. \tag{3.5}$$

Substituting v_t in the indifference condition we obtain the corresponding price sequence

$$p_t = \frac{2}{3} v_0^+ e^{-rt}. \tag{3.6}$$

[15]Since (3.3) holds for all $t > 0$, we can differentiate it twice with respect to t and then combine it with (3.2) to eliminate p_t.

Equilibrium Profit For every given initial cutoff v_0^+, (3.6) describes an equilibrium selling strategy. We now determine the seller-optimal cutoff within this particular class of trigger strategy equilibria. The equilibrium yields the following expected profit for the seller:

$$2v_0^+\left(1-v_0^+\right)p_0+\left(1-v_0^+\right)^2\left(v_0^++\frac{1-v_0^+}{3}\right)+\frac{1}{3}\left(v_0^+\right)^3. \tag{3.7}$$

This expected profit consists of two parts. The first is the expected revenue from the initial auction in which the reserve price is $p_0 = \frac{2}{3}v_0^+$, and buyers with a type higher than v_0^+ participate. The transaction price is p_0 if exactly one buyer has a valuation above v_0^+, which occurs with probability $2v_0^+\left(1-v_0^+\right)$; if both valuations are above v_0^+, which occurs with probability $\left(1-v_0^+\right)^2$, the average transaction price is $v_0^+ + \frac{1-v_0^+}{3}$, that is, the expected value of the lower valuation conditional on both being above v_0^+. The second part is the seller's revenue from the continuation after time $t = 0$. This is realized if both buyers have a valuation below v_0^+, which occurs with probability $(v_0^+)^2$ and by (3.3) the continuation profit is $\frac{1}{3}v_0^+$. The expected profit in (3.7) is maximized by $v_0^+ = \frac{2}{3}$, which implies $p_0 = \frac{4}{9}$.

Comparing with Profit under Full Commitment The profit associated with the equilibrium just constructed can be computed by evaluating (3.7) for $v_0^+ = \frac{2}{3}$. This yields $\frac{31}{81} \approx 0.38$. How does this figure compare with the benchmarks achieved under full commitment and in an efficient auction? The profit is larger than the profit of an efficient auction with zero reserve price, $\Pi^E = \frac{1}{3} \approx 0.33$, but is smaller than the profit of Myerson's optimal auction $\Pi^M = \frac{5}{12} \approx 0.42$.[16] Even though the full commitment profit is not achievable, we have $\frac{0.38-\Pi^E}{\Pi^M-\Pi^E} > 50\%$. Put differently, commitment accounts for less than 50% of the profit increase from running Myerson's optimal auction in an environment with two buyers and uniformly distributed valuations.

Remark In the above construction, we only consider one particular class of trigger strategy equilibria, which is similar in spirit to Ausubel and Deneckere (1989). Is the equilibrium profit $\frac{31}{81}$ the highest profit across all possible equilibria with two bidders? With three or more bidders, we can follow the same heuristics, but it does not lead to an equilibrium because the resulting ODE does not possess a declining solution. In this case, what is the maximal revenue attainable across all equilibria? In particular, are there equilibria that employ strategies more complicated than the simple trigger strategy and attain a rev-

[16] The seller's reserve price in Myerson's optimal auction with full commitment is $\frac{1}{2}$. The optimal reserve price is such that the virtual valuation $v - \frac{1-F(v)}{f(v)}$ equals 0.

enue strictly higher than the revenue from an efficient auction? None of these questions can be addressed by extending the idea of Ausubel and Deneckere (1989). Our main results, obtained through solving an auxiliary mechanism design problem, show that with uniform distribution $\frac{31}{81}$ is indeed the maximal profit for the two bidder case (Theorem 3), and efficient auctions indeed attain the maximal profit in the case of three or more bidders (Theorem 2).

4 Results

This section presents the results of the paper. Our first theorem formalizes our earlier observation that with limited commitment, the revenue from Myerson's optimal auction is not attainable in any perfect Bayesian equilibrium.[17]

Theorem 1. *Suppose Assumption 1 holds. The maximal profit, Π^*, that the seller can achieve in equilibrium as $\Delta \to 0$, is strictly below the seller's profit in Myerson's optimal auction Π^M.*

Note that in order to attain the Myerson's optimal auction profit Π^M, the seller must maintain a constant reserve price in equilibrium. This is impossible because in all equilibria of our game prices must decline to zero. In fact, we prove that, for any fixed $\Delta > 0$, as well as in the limit as $\Delta \to 0$, the maximal profit the seller can attain is strictly below the full commitment profit Π^M.

The main analysis of the paper concerns the characterization of Π^* as well as the set of perfect Bayesian equilibrium payoffs for the seller in the limit as $\Delta \to 0$. The characterization depends on the type distribution and the number of buyers. To state the dependence formally, we define a distribution-specific cutoff $\overline{N}(F)$ for the number of buyers:[18]

$$\overline{N}(F) := 1 + \frac{\sqrt{2+\phi}}{1+\phi}.$$

The first main result shows that if the number of buyers exceeds this cutoff, the maximal equilibrium profit the seller can achieve in the limit is the efficient auction profit. Since the seller can guarantee this profit in any equilibrium (see Lemma 3 below), the set of achievable payoffs contains just Π^E. This result is reminiscent of the finding in Milgrom

[17] Theorem 1 holds without Assumption 1. We focus on the regular case where this assumption holds. Otherwise, Myerson's optimal auction may involve bunching and is not contained in the class of auction formats that we consider.

[18] Recall that $\phi = \lim_{v\to 0} \frac{f'(v)v}{f(v)}$, which exists and is greater than -1 by Assumption 2.

(1987) and McAfee and Vincent (1997), but in contrast to theirs and as we explain later, limited commitment, rather than stationarity, is the driving force here.

Theorem 2. *Suppose Assumptions 1 and 2 hold. If $n > \overline{N}(F)$, then the set of equilibrium profits in the limit is a singleton and $\Pi^* = \Pi_* = \Pi^E$. There exists a sequence of equilibria for which the profit converges to Π^E and the reserve prices for all $t > 0$ converge to 0 as $\Delta \to 0$.*

Depending on the type distribution, the cutoff $\overline{N}(F)$ can take any value above one. For example, if valuations are distributed according to $F(v) = v^k$ with support $[0,1]$ and $k > 0$, we have $\phi = k - 1$ and $\overline{N}(F) = 1 + \sqrt{1+k}/k$. If $k = 1$ we obtain the uniform distribution and $\overline{N}(F) = 1 + \sqrt{2}$. This verifies our claim in Section 3, that with three or more bidders, the seller cannot do better than running an efficient auction if the distribution is uniform. If $k < 1$, the density is unbounded at zero and $\overline{N}(F)$ can be large. Many economic applications, however, focus on distributions with finite densities at 0.[19] For this common class of distributions, $\overline{N}(F)$ is small, as shown in Corollary 1. If the density vanishes at zero but its derivative is not zero, then $\overline{N}(F)$ is even smaller, as shown in Corollary 2 below.

Corollary 1. *Suppose Assumptions 1 and 2 hold. If the density f satisfies $f(0) \in (0, \infty)$, then $\Pi^* = \Pi_* = \Pi^E$ if $n \geq 3$.*

Corollary 2. *Suppose Assumptions 1 and 2 hold. If the density f is twice continuously differentiable at zero, $f(0) = 0$ and $f'(0) \neq 0$, then $\Pi^* = \Pi_* = \Pi^E$ if $n \geq 2$.*

From Theorem 2 (and the complementary Theorem 3.(i) below), we observe that the optimality of the efficient auction in the limit only depends on the lower tail of the distribution. The intuition is as follows. At any time t, the seller's posterior is a truncation from above of the original distribution. Therefore, the tail of the distribution determines the set of equilibria in subgames which start after sufficiently many periods. Suppose the tail of the distribution allows multiple equilibria in every subgame starting in period $t + \Delta$. Then, there are also multiple equilibria in any subgame starting at t. In contrast, if the tail of

[19] This includes all distributions with monotone virtual valuation—a standard assumption in mechanism design which is slightly stronger than our Assumption 1. To see this, we first argue that $\lim_{v \to 0} f(v)\, v = 0$. Suppose by contradiction $\lim_{v \to 0} f(v)\, v = z > 0$. Then we must have $f(v) > z/v$ for a small neighborhood $(0, \varepsilon)$, which implies that $\int_0^\varepsilon f(v)\, dv > \int_0^\varepsilon \frac{z}{v} dv = \infty$, a contradiction to the fact that f is a density. Next we note that $\lim_{v \to 0} f(v) = \infty$ implies that $\lim_{v \to 0} J(v) = 0$. If $J'(v) \geq 0$ for all v, $\lim_{v \to 0} J(v) = 0$ implies $J(v) \geq 0$ for all v and hence $F(v) \geq 1 - v f(v)$. Taking the limit as $v \to 0$ and using $\lim_{v \to 0} v f(v) = 0$, we can deduce from this inequality that $F(0) = 1$ which contradicts our assumption that F has a density. Hence, $\lim_{v \to 0} f(v) < \infty$. We thank a referee for pointing this out to us.

the distribution pins down a unique continuation equilibrium for all possible histories after sufficiently many periods, then there is a unique equilibrium in the whole game. Therefore, the degeneracy of the equilibrium set hinges on properties of the tail of the distribution.

If $n < \overline{N}(F)$, the efficient auction no longer attains the highest equilibrium revenue. We construct a sequence of equilibria that achieves $\Pi^* > \Pi^E$ and characterize the entire set of limiting profits that the seller can obtain in equilibrium. To do this, we need to introduce some notation. We define a function $g : (0,1] \to \mathbb{R}$ that will be used to characterize the limiting outcome (in terms of v_t) that achieves Π^*:

$$g(x) = \frac{f'(x)}{f(x)} - \frac{\left[x\,(F(x))^{n-1} - 2\int_0^x (F(v))^{n-1}\,dv\right] f(x)}{(n-1)\int_0^x [F(x) - F(v)]\,(F(v))^{n-2} f(v)\,v\,dv}.$$

Theorem 3. *Suppose Assumptions 1, 2, and 3 hold, and $n < \overline{N}(F)$.*

(i) $\Pi^* > \Pi_* = \Pi^E$.

If in addition, Assumption 4 holds:

(ii) $\Pi^* > \Pi_* = \Pi^E$ *is achieved by a sequence of equilibria with positive reserve prices in which the buyers' equilibrium cutoff paths converge to a cutoff path that starts with some $v_0^+ > 0$, and it is given by the unique solution of the differential equation*

$$\dot{v}_t = -\int_0^{v_t} r e^{\int_v^{v_t} g(x)dx} dv. \tag{4.1}$$

The corresponding path of reserve prices is given by[20]

$$p_t = v_t + \int_t^{\infty} e^{-r(s-t)} \left(\frac{F(v_s)}{F(v_t)}\right)^{n-1} \dot{v}_s ds, \quad \forall t > 0. \tag{4.2}$$

(iii) Any $\Pi \in \left[\Pi^E, \Pi^*\right]$ is a limit of a sequence of equilibrium payoffs as $\Delta \to 0$.

Assumption 4 is used in parts (ii) and (iii) of Theorem 3 to show that the seller's incentive constraint must become binding in the limit as $\Delta \to 0$ in order to achieve Π^*.[21] In particular, this implies that Π^* is achieved by an initial auction followed by a continuously declining

[20] The initial price at $t = 0$ is given by $p_0 = v_0^+ + \int_0^\infty e^{-rs} \left(F(v_s)/F(v_0^+)\right)^{n-1} \dot{v}_s ds$.

[21] For part (i), Assumption 4 is not needed because it suffices to construct a limiting outcome that achieves a profit greater than Π^E but not necessarily equal to Π^*.

reserve price that satisfies the ODE (4.1).[22] Without Assumption 4, we cannot rule out that the reserve price jumps down at times $t > 0$, so that a positive measure of types is induced to participate in an auction at the same point in time. It can be verified that in the example of Section 3, equations (4.1) and (4.2) pin down the limiting equilibrium that attains the maximal profit $\Pi^* = \frac{31}{81}$.

Next we discuss the logic behind the results in Theorems 2 and 3. Theorem 2 can be interpreted as a Coase conjecture result without stationarity restriction. A related Coase conjecture result is obtained in Milgrom (1987) and McAfee and Vincent (1997), but their result is entirely driven by their stationarity restriction. In stationary equilibria, all bidders follow stationary bidding strategies which can be interpreted as a *demand curve* faced by the seller. The seller would like to collect the *surplus* below the demand curve as quickly as possible. As $\Delta \to 0$, she can collect the whole surplus by setting more and more finely spaced reserve prices in shorter and shorter intervals. Prices must therefore decline to zero immediately which implies that the demand curve collapses to zero as well, and the Coase conjecture follows. This logic works independent of the type distribution and the number of buyers but crucially relies on stationarity.[23] In contrast, Theorem 2 imposes no stationarity restriction, and shows that limited commitment alone can obtain the efficient auction as the unique limit equilibrium outcome if the number of bidders is above the distribution-specific cutoff. Therefore, Theorem 2 helps clarify the role of limited commitment in the auction setting.

We give an intuitive explanation why the efficient auction is revenue-maximizing if the number of buyers is above the distribution-specific cutoff, and otherwise we obtain an intermediate result between the folk theorem of Ausubel and Deneckere (1989) for ($n = 1$) and the Coase conjecture. As shown by Lemma 3 below, the seller in the auction setting can guarantee herself a strictly positive profit because she can always run an efficient auction immediately. Hence, at any point in time, to deter the seller from running the efficient auction, the continuation profit from screening must exceed the profit from immediately running the efficient auction. This creates a tension between screening types optimally from the perspectives of any two times t and s.[24] This tension can be resolved more easily and a declining

[22] We explain in Section A.1 how (4.1) is obtained from the seller's incentive constraint.

[23] Proposition 2, which establishes the Coase conjecture for stationary equilibria in our auction setting only requires Assumption 3.

[24] To see this, note that to induce the seller to continue screening at time t, one can choose a price path $(p_\tau)_{\tau > t}$ to induce high types to trade early and low types to delay trades. At a later time $s > t$ after most high types have traded, however, the seller must speed up the trade with some of these low types to generate sufficiently high continuation profit in order for her to resist the temption to run the efficient auction, creating a conflict in inducing her to continue screening at the earlier time t.

price path can be part of an equilibrium, if the number of buyers is small $\left(n < \overline{N}(F)\right)$ so that running the efficient auction is less attractive relative to continued screening. Otherwise, active screening is strictly dominated by running the efficient auction immediately, so the efficient auction is the unique limit equilibrium outcome.

The comparison between the profit from an efficient auction and the potential benefits from screening can also help understand the gap case, as analyzed by McAfee and Vincent (1997), where the buyers' type distribution has support $[\varepsilon, 1]$. By posting price $p_t = \varepsilon$, the seller can guarantee herself a profit $\varepsilon > 0$, even with one buyer. In contrast to the no-gap auction case where the lower bound on the seller's profit at time t (i.e., the profit from running the efficient auction at time t) goes to zero as $v_t \to \varepsilon$, here the profit bound ε is a constant independent of v_t. In fact, for v_t sufficiently close to ε, the profit attainable by setting $p_t = \varepsilon$ coincides with the full commitment profit. As a result, the game ends in finite time which implies that all equilibria must be stationary.[25] Hence, in the gap case, the Coase conjecture directly follows from stationarity.

5 Methodology and Overview of Proofs

Our strategy to characterize Π^*, the corresponding limit price path, and the set of limit equilibrium profits for the seller, is to analyze an auxiliary dynamic mechanism design problem. To formulate the problem, we identify basic properties of equilibria of the discrete time game (Section 5.1). These properties are necessary conditions for equilibrium outcomes. We then formulate the same restrictions in continuous time and use them to define the feasible set of mechanisms in the dynamic mechanism design problem (Section 5.2). Necessity of the constraints implies that the value of the auxiliary problem is an upper bound for Π^*. To establish sufficiency, we show that the optimal value of the auxiliary problem is attained by a sequence of discrete time equilibria as period length goes to zero. Therefore, the optimal value of the auxiliary problem is exactly the maximal profit attainable in any equilibrium in the continuous time limit.

[25] In the gap-case where the last period is endogenous, as well as in a game with an exogenous last period, the equilibrium can be found by backward induction. This implies that it is essentially unique. In both cases reputational equilibria are ruled out by uniqueness.

5.1 Equilibrium Properties

In any equilibrium of the discrete time game, all buyers play pure strategies that are characterized by history-dependent cutoffs. This is captured by the following Lemma which establishes the "skimming property," an auction analog of a result by Fudenberg, Levine, and Tirole (1985). Its proof is standard and thus omitted.

Lemma 1 (Skimming Property)**.** *Let $(p, b) \in \mathcal{E}(\Delta)$. Then, for each $t = 0, \Delta, 2\Delta, \ldots$, there exists a function $\beta_t : H_t \times [0, 1] \to [0, 1]$ such that every bidder with valuation above $\beta_t(h_t, p_t)$ places a valid bid and every bidder with valuation below $\beta_t(h_t, p_t)$ waits if the seller announces reserve price p_t at history h_t.*

The next lemma shows that randomization on the equilibrium path is not necessary to attain the maximal profit.

Lemma 2. *For every equilibrium $(p, b) \in \mathcal{E}(\Delta)$, there exists an equilibrium $(p', b') \in \mathcal{E}(\Delta)$ in which the seller does not randomize on the equilibrium path and achieves a profit $\Pi^\Delta(p', b') \geq \Pi^\Delta(p, b)$.*

Lemma 1 implies that at any history, the posterior of the seller is given by a truncation of the prior. Lemmas 1 and 2 together imply that for the characterization of Π^*, we can restrict attention to equilibrium allocation rules which are deterministic (up to tie-breaking).[26] Symmetric deterministic equilibrium allocation rules can be described in terms of a trading time function $T : [0, 1] \to \{0, \Delta, 2\Delta, \ldots\}$ which must be non-increasing because of Lemma 1. Given that buyers bid truthfully in a second-price auction, in any symmetric equilibrium the object will be allocated at time $T(v^{(n)})$, to the bidder with the highest valuation.

The last lemma in this section shows that the seller can ensure a continuation profit no smaller than the profit of an efficient auction, even though running an efficient auction is not a part of an equilibrium.

Lemma 3. *Fix any equilibrium $(p, b) \in \mathcal{E}(\Delta)$ and any history h_t. If the seller announces the reserve price $p_t = 0$ at h_t (this may not be part of an equilibrium strategy), then every bidder bids his true value and the game ends.*

Lemma 3 provides a lower bound for the seller's payoff on and off the equilibrium path which provides a constraint for continuation payoffs in the auxiliary problem introduced below. It also follows from Lemma 3 that $\Pi_* \geq \Pi^E$. See the Supplemental Material for proofs of Lemmas 2 and 3.

[26] The proof of our main results shows that this restriction is also without loss for the *set* of limit profits achievable for the seller.

5.2 The Auxiliary Mechanism Design Problem

In the auction context, limited commitment invalidates the full commitment solution as a target for equilibrium construction, so we have to first find the maximal equilibrium profit in order to characterize the entire set of equilibrium profits for the seller. In this subsection, we set up the auxiliary mechanism design problem with full commitment which we use to chacterize the maximal profit, and briefly explain why solving the auxiliary problem constitutes the crucial step in proving the main results.

5.2.1 Mechanisms

The auxiliary mechanism design problem is formulated in continuous time and assumes the seller has full commitment power. Buyers participate in a direct mechanism and make a single report of their valuations at time zero. The mechanism awards the object to the buyer with the highest reported type (up to tie breaking). It specifies a deterministic and non-increasing trading time function $T : [0, 1] \to [0, \infty]$. If the mechanism awards the object to buyer i, then the allocation takes place at time $T(v^i)$. This is motivated by Lemmas 1 and 2. Moreover, the mechanism specifies a payment for the winning bidder.[27]

The discounted trading probability of a bidder with type v is $e^{-rT(v)}$ if he is the highest bidder and zero otherwise. The (interim) expected discounted winning probability of a buyer is thus $\Pr\{v^i = \max_j v^j\}\, e^{-rT(v)}$, and this is non-decreasing since T is non-increasing. Therefore, any non-increasing trading time function is implementable, and following standard arguments, individual rationality and incentive compatibility constraints for the buyers can be used to express the seller's profit as

$$\int_0^1 J(v)\, e^{-rT(v)} F^{(n)}(v). \tag{5.1}$$

Let us define cutoff types as

$$v_t := \sup\{v \mid T(v) \geq t\}.$$

v_t is the highest type that does not trade before time t. Since all buyers with types $v > v_t$ trade before t, the posterior distribution at t, conditional on the event that the object has not yet been allocated, is given by the truncated distribution $F(v|v \leq v_t)$. Therefore, we call v_t the *posterior at time* t. We denote the posterior distribution functions and the virtual

[27]We restrict attention to mechanisms that only require payments from the winning bidder as is the case for second-price auctions. This can be generalized easily to other auction formats discussed in Remark 1 in Section 2.

valuation for the posterior at time t by

$$F_t(v) := \frac{F(v)}{F(v_t)}, \qquad F_t^{(n)}(v) := \frac{F^n(v)}{F^n(v_t)},$$

and

$$J_t(v) := v - \frac{F(v_t|v \le v_t) - F(v|v \le v_t)}{f(v|v \le v_t)} = v - \frac{F(v_t) - F(v)}{f(v)}.$$

Generally, v_t is continuous from the left, and since it is non-increasing, the right limit exists everywhere. We will denote the right limit at t by

$$v_t^+ := \lim_{s \searrow t} v_s.$$

For each t, v_t^+ is the highest type in the posterior after time t if the object is not yet sold.

Under Assumption 1, if the seller has full commitment, the dynamic mechanism design problem of maximizing (5.1) without further constraints, reduces to the static problem of Myerson (1981). The optimal solution is to allocate to the buyer with the highest valuation if his virtual valuation is non-negative, and otherwise to withhold the object. Formally, in terms of trading times, Myerson's solution is given by

$$T^M(v) := \begin{cases} 0 & \text{if } J(v) \ge 0, \\ \infty & \text{if } J(v) < 0. \end{cases} \tag{5.2}$$

5.2.2 Payoff Floor Constraint

To obtain an auxiliary problem that captures the seller's incentives under limited commitment, we add an additional constraint. Motivated by Lemma 3 we assume that the continuation payoff of the seller must be bounded below by the revenue of an efficient auction for the given posterior at each point in time. To state this "payoff floor constraint" formally, we denote the revenue from an efficient auction for the posterior v_t as

$$\Pi^E(v_t) = \frac{1}{F^{(n)}(v_t)} \int_0^{v_t} J_t(x) dF^{(n)}(x).$$

The seller's continuation payoff from the dynamic mechanism at time t can be formulated as

$$\frac{1}{F^{(n)}(v_t)} \int_0^{v_t} e^{-r(T(x)-t)} J_t(x) dF^{(n)}(x).$$

Therefore, the payoff floor constraint (PF) is given by (where we have dropped the term $1/F^{(n)}(v_t)$ on both sides):

$$\int_0^{v_t} e^{-r(T(x)-t)} J_t(x) dF^{(n)}(x) \geq \int_0^{v_t} J_t(x) dF^{(n)}(x), \text{ for all } t \geq 0. \tag{5.3}$$

The payoff floor constraint introduces a dynamic element into the auxiliary problem that distinguishes it from a standard static mechanism design problem under full commitment.

5.2.3 Auxiliary Problem

To summarize, we can formulate the auxiliary problem as the following dynamic mechanism design problem:

$$\sup_{T:[0,1]\to[0,\infty]} \int_0^1 e^{-rT(x)} J(x) dF^{(n)}(x) \tag{5.4}$$

$$\text{s.t.} \quad \begin{array}{l} \text{IC: } T \text{ is non-increasing,} \\ \text{PF: } \displaystyle\int_0^{v_t} e^{-r(T(x)-t)} J_t(x) dF^{(n)}(x) \geq \int_0^{v_t} J_t(x) dF^{(n)}(x), \forall t \geq 0. \end{array}$$

We call any $T : [0, 1] \to [0, \infty]$ that satisfies (IC) and (PF) a *feasible solution* of the auxiliary problem. We denote the value of the auxiliary problem by V and standard techniques can be used to show that an optimal solution exists (see Proposition 1 in Appendix A).

The payoff floor constraint rules out a deviation by the seller to an efficient auction, which is a necessary condition for an equilibrium. Therefore, V is an upper bound for the seller's maximal profit Π^*, which is formally proved in Proposition 3 in Appendix A. We then show that V is achievable by a sequence of discrete-time equilibria as $\Delta \to 0$. If $V = \Pi^E$, this follows directly from the existence of stationary equilibria (Proposition 2.(i) in Appendix A). If $V > \Pi^E$ the construction uses the simple trigger strategy with stationary equilibria as off-path punishment. This is possible because the profit of stationary equilibria converges to the right-hand side of the payoff floor constraint as $\Delta \to \infty$ (see Proposition 2.(ii)). Therefore, the payoff floor constraint exactly captures limited commitment and the optimal value of the auxiliary problem is exactly the maximum revenue attainable in any equilibrium as the seller's commitment ability vanishes.

In order to characterize Π^*, the revenue maximizing cutoffs and reserve prices, and the set of limiting profits achievable for the seller, it is adequate to solve the auxiliary problem. Theorem 1 directly follows from Proposition 3 because Myerson's optimal auction is not a feasible solution to the auxiliary problem. To prove Theorems 2 and 3, we start by constructing a solution candidate to the auxiliary problem by assuming that the payoff floor constraint binds for all $t > 0$. This yields the ODE in (4.1). We show that if $n > \overline{N}(F)$, the solution to this ODE is increasing, and therefore, we cannot obtain a feasible (i.e. decreasing) solution from the binding payoff floor constraint. The crucial step to obtain Theorem 2 is to show that no other feasible solution except the efficient auction exists if the binding payoff floor constraint yields an increasing cutoff path. For Theorem 3, we show that the binding payoff floor constraint yields a decreasing (and thus feasible) solution if $n < \overline{N}(F)$. Moreover, we show that under Assumption 4 the payoff floor must be binding at the optimal solution to the auxiliary problem. The formal development of these steps can be found in Appendix A.

6 Concluding Remarks

In this paper we have studied the role of commitment power in auctions where the seller cannot commit to future reserve prices. Our analysis draws insights from the bargaining literature, and the auction and mechanism design literature. We conclude the paper by discussing our modeling assumptions and possible extensions.

The Role of Assumption 1. Under Assumption 1, Myerson (1981) shows that the maximal full-commitment profit is attained by a standard auction with a reserve price equal to $\underline{v}$.[28] It excludes low types below the reserve, while allocating the object immediately to the buyer with the highest valuation that is above the reserve. The auction mechanisms we consider in the paper also allocate the object to the buyer with the highest valuation, but due to limited commitment, exclusion is not possible, and delaying the allocation is the only way to screen different buyer types. Therefore, under Assumption 1, the difference between our limited commitment solution and Myerson's optimal auction solution can be attributed to the different assumptions about the seller's commitment. Indeed, our auxiliary mechanism design problem without the payoff floor constraint is a problem with full commitment, and Myerson's optimal auction solves this problem under Assumption 1. Hence, the payoff floor constraint captures the key aspect of limited commitment. A different way to look at the issue of commitment is through the original discrete-time problem. As we have explained previously, if Δ approaches infinity, our problem is becomes a full-commitment problem, and Δ can be understood as the ability to commit, as in the classic durable goods monopoly literature.

[28] Myerson's regularity condition is slightly stronger than our Assumption 1. We formulate the weaker assumption to allow for distributions with unbounded density at zero (see also footnote 19).

If Assumption 1 fails, the connection between our model of limited commitment and Myerson's optimal auction with full commitment is not as straightforward. Myerson shows that ironing must be used and the optimal auction may randomize the allocation between buyers with different types. Consequently, with full commitment, there is an inefficiency in addition to the exclusion of low types. The class of auction mechanisms we consider in the paper, however, does not allow allocative inefficiency through ironing. Therefore, without Assumption 1, we cannot fully separate the effect of limited commitment on the seller's payoff from the effect of our restriction to the particular class of auction mechanisms.

There are two ways to proceed with the analysis if Assumption 1 fails. First, we could try to extend the class of mechanisms. For example, we can allow the seller to choose in each period between an auction with a reserve price and a posted price with the possibility of rationing. The corresponding auxiliary problem without payoff floor constraint would allow to implement Myerson's optimal auction. While this model retains the crucial skimming property, it is not trivial to analyze a generalized version of the auxiliary problem. Second, by restricting attention to the same class of mechanisms currently considered in the paper, we can still ask the question of whether our results for reserve price auctions go through, and whether we can still go beyond the existing insights of Milgrom (1987) and McAfee and Vincent (1999) if we drop Assumption 1. We have a positive answer to this. Assumption 1 is imposed in order to obtain a clean modeling framework, but it is not essential for any of the proofs.

Symmetry Restriction. Throughout the paper, we have restricted attention to buyer-symmetric equilibria. Symmetry plays a crucial role in establishing the revenue equivalence theorem in auction theory, and is likewise important in extending our analysis of second-price auctions to the more general class of allocative efficient auctions. Besides this immediate consequence, we now highlight other roles played by the symmetry assumption. If we allow for asymmetric equilibria, we can formulate an asymmetric auxiliary problem in terms of a trading time function (or a sequence of cutoffs) for each buyer. Since the seller can only choose a single price in each period, however, the set of implementable cutoff sequences for a given buyer depends on the cutoff sequences chosen for the other buyers. Therefore, the asymmetric auxiliary problem requires additional constraints which are quite complex and not very tractable. A more fundamental problem for a tractable specification of the auxiliary problem arises because we do not know how to extend the proof of Lemma 2 to asymmetric equilibria.[29] Consequently, we cannot restrict attention to deterministic allocation rules.

[29] In the proof for the symmetric case, for any (possibly mixed) equilibrium, we select the sequence of (symmetric) cutoffs implemented along one particular on-path history. Since every symmetric sequence of cutoffs

Finally, symmetry also helps to rule out that buyers play dominated strategies in second-price auctions, which is a standard assumption.[30] In light of these issues, it seems that the complications involved in studying asymmetric equilibria are on par with the complications that arise when analyzing general mechanisms. We believe that the analysis of general mechanisms is a fruitful direction for future research but is beyond the scope of the paper.

Modeling Limited Commitment. Our way of modeling limited commitment assumes that the seller can commit to the terms of trade within a single period: if $\Delta = \infty$, there is full commitment; as $\Delta \to 0$, the seller's commitment power vanishes. This approach is taken by Milgrom (1987) and McAfee and Vincent (1997).

An alternative modeling approach is to assume that the seller's opportunity of running an additional auction is uncertain. This can be cast into a continuous-time framework as follows. There is a Poisson arrival of auction opportunities, with constant arrival rate λ. An auction can only be held at time $t = 0$ or when there is an arrival. If $\lambda = 0$, there is full commitment; if $\lambda \to \infty$, the commitment power vanishes. This model is similar to ours except that the period length Δ is random, but $\Delta \to 0$ in distribution as $\lambda \to \infty$.

Another way to formulate the problem of limited commitment is to allow long-term contracts and renegotiation (see Hart and Tirole, 1988; Strulovici, 2013, and references therein). In our setup with multiple bidders, however, modeling renegotiation introduces new conceptual issues, such as the protocol of multiple-person bargaining and signaling in the renegotiation phase.

Unknown Number of Bidders. We assume that the seller knows the number of serious bidders, and normalize the seller's commonly known reservation value to be 0. This is a natural assumption because a bidder who knows that his value is below the seller's reservation value will not obtain the object in any case and will not show up in an auction. A natural research question is what happens when the seller is uncertain about the number of serious buyers. With full commitment, the problem of an uncertain number of bidders has first been studied by McAfee and McMillan (1987). Without commitment, a possible modeling approach is to assume that there are n bidders whose values are distributed over

is implementable by some sequence of reserve prices, we are able to construct a new equilibrium without on-path randomization and weakly higher profits. With asymmetric cutoffs, this is no longer possible because the cutoffs implemented along a particular history may not be implementable by a single deterministic price sequence.

[30] For $n > 2$, Blume and Heidhues (2004) show that the second-price auction has a unique equilibrium if the seller uses a non-trivial reserve price. Therefore, symmetry is not needed to rule out low-profit equilibria if $n > 2$. By posting a reserve price close to zero, the seller can end the game with probability arbitrarily close to one and guarantee herself a profit arbitrarily close to the profit of an efficient auction. This implies that the lower bound for the seller's equilibrium payoff that we obtain in Lemma 3 is independent of the symmetry assumption if there are at least three buyers.

[0, 1], but the seller's reservation value c is interior. In this case, the seller is uncertain about the number of bidders whose values are above c; indeed, it is possible that no bidder has a value above c. Over time, the seller will update her belief about the number of serious bidders and their valuations. Eventually, the seller will believe that the number of bidders is small, and hence the seller will slow down the decline of the reserve price, which can be used to support an equilibrium that fares better than an efficient auction.

A Appendix

In this appendix, we sketch the key steps in characterizing the optimal solutions to the auxiliary problem, which will form the basis of our proofs of Theorems 1–3. Except for the proof of Proposition 2 (existence of stationary equilibria and uniform Coase conjecture) which is in Section C, all other proofs omitted from this appendix are collected in Section B of the Supplemental Material where one can also find several omitted intermediate steps in solving the auxiliary problem. Section D of the Supplemental Material constructs equilibria that approximate the solution to binding payoff floor constraint and proves Proposition 6 which is used in the proof of Theorem 3.

A.1 Analysis of the Auxiliary Problem

A.1.1 Basic Results

For $n = 1$, the case of a single buyer, the right-hand side of the payoff floor constraint is zero, and the optimal solution is T^M.[31] For $n \geq 2$ this is not the case, as shown in the following lemma.

Lemma 4. *For any T in the feasible set of the auxiliary problem, $T(v) < \infty$ for all $v > 0$ and $\lim_{t\to\infty} v_t = 0$.*

Proof. Suppose by contradiction that T is feasible but $T(v) = \infty$ for some $v > 0$. Since T is non-increasing, there exists $w \in (0, 1)$ such that $T(v) = \infty$ for all $v \in [0, w)$ and $T(v) < \infty$ for all $v \in (w, 1]$. The left-hand side of the payoff floor constraint can be rewritten as, for all $t < \infty$,

$$\int_0^{v_t} e^{-r(T(x)-t)} J_t(x) dF^{(n)}(x) = \int_w^{v_t} e^{-r(T(x)-t)} J_t(x) dF^{(n)}(x).$$

[31] This also implies the folk-theorem obtained by Ausubel and Deneckere (1989).

Since $T(v) < \infty$ for all $v \in (w, 1]$, we have $v_t \to w$ as $t \to \infty$. Hence, as $t \to \infty$, the limit of the left-hand side is zero:

$$\lim_{t\to\infty} \int_w^{v_t} e^{-r(T(x)-t)} J_t(x) dF^{(n)}(x) = 0.$$

The limit of right-hand side of the payoff floor constraint as $t \to \infty$, however, is strictly positive:

$$\lim_{t\to\infty} \int_0^{v_t} J_t(x) dF^{(n)}(x) = \int_0^w \left(x - \frac{F(w) - F(x)}{f(x)} \right) dF^{(n)}(x) > 0.$$

Therefore, the payoff floor constraint must be violated for sufficiently large t, which contradicts the feasibility of T. □

The following existence result is standard, and its proof (as well as all other omitted proofs) can be found in the Supplemental Material.

Proposition 1. *An optimal solution to the auxiliary problem exists.*

A.1.2 Optimal Value as Equilibrium Revenue Upper Bound

Based on Ausubel and Deneckere (1989) we start by showing existence of stationary equilibria, i.e., equilibria with stationary buyer-strategies that only depend on the valuation and the current reserve price. We also generalize the uniform Coase conjecture for stationary equilibria to the auction setting.

Proposition 2. *(i) (Existence) A stationary equilibrium exists for every $r > 0$ and $\Delta > 0$.*

(ii) (Uniform Coase Conjecture) Suppose Assumption 3 holds. For every $\varepsilon > 0$, there exists $\Delta_\varepsilon > 0$ such that for all $\Delta < \Delta_\varepsilon$, all $x \in [0, 1]$, and every symmetric stationary equilibrium (p, b) of the game with period length Δ and a truncated distribution $F(v|v \leq x)$ on $[0, x]$, the seller's profit associated with this equilibrium, $\Pi^{\Delta}(p, b|x)$, is bounded above by $(1 + \varepsilon) \Pi^E(x)$, where $\Pi^E(x)$ is the seller's profit from the efficient auction under this truncated distribution.

The second part of the proposition shows that the seller's profit in every symmetric stationary equilibrium converges to the profit of the efficient auction.[32] Uniform convergence,

[32]Notice that in contrast to the Coase conjecture for one buyer, Proposition 2.(ii) does not show that the initial reserve price p_0 converges to zero. This is in fact not the case in the auction setting as was noted by McAfee and Vincent (1997). However, reserve prices for $t > 0$ converge to zero which is sufficient for the convergence of equilibrium profits to the profit of an efficient auction—the counterpart of the Coase conjecture in the auction setting.

in the sense that $\Pi^{\Delta}(p, b|x) / \Pi^E(x) \to 1$ uniformly for all $x \in (0,1]$, will be used in the construction of trigger strategy equilibria for Theorem 3.

Clearly, the lower bound of the seller's profit for all equilibria is achievable by $T^E(v) \equiv 0$. This corresponds to a second-price auction with reserve price $p_t = 0$ at time $t = 0$, and $T^E(v) \equiv 0$ implies $v_t = 0$ for all $t > 0$. Therefore, the payoff floor constraint is trivially satisfied for both $t > 0$ and $t = 0$. The following result shows that the optimal value of the auxiliary problem is an upper bound for all equilibrium revenues in the original game.

Proposition 3. *Let (Δ_m) be a decreasing sequence with $\Delta_m \searrow 0$, and let (p_m, b_m) be a sequence of equilibria in which the seller does not randomize on the equilibrium path. Then $\limsup_{m\to\infty} \Pi^{\Delta_m}(p_m, b_m) \in [\Pi^E, V]$. In particular $\Pi^* \leq V$.*

Proof. We first define an ε-relaxed continuous-time auxiliary problem. We replace the payoff floor constraint by

$$\int_0^{v_t} e^{-r\,(T(x)-t)} J_t(v) dF_t^{(n)}(v) \geq (1-\varepsilon)\Pi^E(v_t).$$

By the maximum theorem, the value of this problem, which we denote by V_ε, is continuous in ε—that is, $\lim_{\varepsilon\to 0} V_\varepsilon = V$.

Next, we formulate a discrete version of the auxiliary problem. For given Δ, the feasible set of this problem is given by

$$T : [0,1] \to \{0, \Delta, 2\Delta, \ldots\} \text{ non-increasing,}$$

$$\text{and} \quad \int_0^{v_{k\Delta}} e^{-r\,(T(x)-k\Delta)} J_{k\Delta}(v) dF_{k\Delta}^{(n)}(v) \geq \Pi^E(v_{k\Delta}) \qquad \forall k \in \mathbb{N}.$$

We denote the value of this problem by $V(\Delta)$. Let $\mathcal{E}^d(\Delta) \subset \mathcal{E}(\Delta)$ denote the set of equilibria in which the seller does not randomize on the equilibrium path. The first constraint is clearly satisfied for outcomes of any equilibrium $\mathcal{E}^d(\Delta)$. The second constraint requires that in each period, the seller's continuation profit on the equilibrium path exceeds the revenue from an efficient auction given the current posterior. By Lemma 3, this is a necessary condition for an equilibrium. Therefore, the seller's expected revenue in any equilibrium $(p, b) \in \mathcal{E}^d(\Delta)$ cannot exceed $V(\Delta)$. Moreover, for given ε, the feasible set of the discrete auxiliary problem is contained in the feasible set of the ε-relaxed continuous-time auxiliary problem if Δ is sufficiently small. Formally, we have:

Claim: Let $\varepsilon > 0$ and $\Delta_\varepsilon = -\ln(1-\varepsilon)/r$. For all $\Delta < \Delta_\varepsilon$ we have

$$\sup_{(p,b)\in\mathcal{E}^d(\Delta)} \Pi^{\Delta}(p, b) \leq V(\Delta) \leq V_\varepsilon.$$

Proof of the claim: The first inequality has been shown in the text above. For the second, let T^Δ be an element of the feasible set of the discrete auxiliary problem for $\Delta \leq \Delta_\varepsilon$. Let v_t^Δ be the corresponding cutoff path. Note that for $t \in (k\Delta, (k+1)\Delta]$ we have $v_t^\Delta = v_{(k+1)\Delta}^\Delta$ and hence

$$\begin{aligned}
&\int_0^{v_t^\Delta} e^{-r\,(T^\Delta(v)-t)} J_t(v) n(F(v))^{n-1} f(v) dv \\
=& e^{-r((k+1)\Delta - t)} \int_0^{v_{(k+1)\Delta}^\Delta} e^{-r\,(T^\Delta(v)-(k+1)\Delta)} J_{(k+1)\Delta}(v) n(F(v))^{n-1} f(v) dv \\
\geq& e^{-r\Delta} \int_0^{v_{(k+1)\Delta}^\Delta} e^{-r\,(T^\Delta(v)-(k+1)\Delta)} J_{(k+1)\Delta}(v) n(F(v))^{n-1} f(v) dv \\
\geq& e^{-r\Delta} \Pi^E(v_{(k+1)\Delta}^\Delta) \\
=& e^{-r\Delta} \Pi^E(v_t^\Delta) \\
\geq& (1-\varepsilon) \Pi^E(v_t^\Delta).
\end{aligned}$$

The first inequality holds because $t \geq k\Delta$, the second inequality follows from the payoff floor constraint of the discretized auxiliary problem, and the last inequality holds because $\Delta \leq \Delta_\varepsilon$. Therefore, T^Δ is a feasible solution for the ε-relaxed continuous time auxiliary problem, and hence $V(\Delta) \leq V_\varepsilon$ if $\Delta < \Delta_\varepsilon$. Thus the claim is proved.

To complete the proof for Proposition 3, it suffices to show $\Pi^* \leq V$, which follows directly from Lemma 2 and the claim above:

$$\Pi^* = \limsup_{\Delta \to 0} \sup_{(p,b) \in \mathcal{E}^d(\Delta)} \Pi^\Delta(p, b) \leq \lim_{\varepsilon \to 0} V_\varepsilon = V.$$

□

A.1.3 Implementing Decreasing Trading Time through Price Path

Any non-increasing trading time function T (with cutoffs v_t) can be implemented, that is, there exists a sequence of reserve prices p_t such that for all t, all types above v_t^+ strictly prefer to bid before or at time t, all lower types strictly prefer to wait, and type v_t^+ is indifferent between buying immediately at price p_t and waiting.[33] This price sequence can be obtained from the envelope formula for the buyers' payoff.

[33] Note that v_t^+ is the infimum of all types that trade at time t. Therefore, if the reserve price at time t is p_t, the buyer with valuation v_t^+ will pay price p_t if she makes a truthful bid at time t and this bid wins.

Lemma 5. *Let $T : [0,1] \to [0,\infty]$ be non-increasing and $(v_t)_{t\in\mathbb{R}}$ the corresponding sequence of cutoffs. Then the following sequence of prices implements $(v_t)_{t\in\mathbb{R}}$:*

$$p_t = v_t^+ - \int_0^{v_t^+} e^{-r(T(v)-t)} \left(\frac{F(v)}{F\left(v_t^+\right)}\right)^{n-1} dv. \tag{A.1}$$

If v_t is differentiable, we have $v_t^+ = v_t$, and obtain equation (4.2)

$$p_t = v_t + \int_t^{\infty} e^{-r(s-t)} \left(\frac{F(v_s)}{F(v_t)}\right)^{n-1} \dot{v}_s ds.$$

Proof. First consider period t such that $T(v_t^+) = t$. The expected discounted payoff of any v with $T(v) = t$ is given by

$$\begin{aligned} U^i(v) =& e^{-rt} \int_{v_t^+}^{v} (v-x) dF^{n-1}(x) + F\left(v_t^+\right)^{n-1} e^{-rt} (v - p_t), \\ =& e^{-rt} \int_{v_t^+}^{v} F^{n-1}(x)\, dx + F\left(v_t^+\right)^{n-1} e^{-rt} \left(v_t^+ - p_t\right). \end{aligned}$$

The expected discounted winning probability of any type x is

$$Q^i(x) = F(x)^{n-1} e^{-rT(x)}.$$

Inserting this into the payoff equivalence formula, we obtain

$$\begin{aligned} e^{-rt} \int_{v_t^+}^{v} F^{n-1}(x)\, dx + F\left(v_t^+\right)^{n-1} e^{-rt} \left(v_t^+ - p_t\right) &= \int_0^v e^{-rT(x)} (F(x))^{n-1} dx \\ \iff F\left(v_t^+\right)^{n-1} e^{-rt} \left(v_t^+ - p_t\right) &= \int_0^{v_t^+} e^{-rT(x)} (F(x))^{n-1} dx \end{aligned}$$

which can be rearranged to (A.1). Next, for $T(v_t^+) > t$, by the same envelope formula, (A.1) ensures that the marginal type is indifferent between bidding at time t and waiting until time $T(v_t^+)$. □

A.1.4 Characterizing Optimal Solutions

We now prove intermediate results which are useful in characterizing optimal solutions to the auxiliary problem and the set of feasible profits. In Section B.6 in the Supplemental material we provide proofs and in some cases more general statements for these results that are not used in the analysis of the auxiliary problem but are only needed for the equilibrium approximation. To understand the main argument that leads to the characterization of V, the following intermediate results are sufficient.

First, we show that the efficient auction (T^E) is optimal if and only if it is the only feasible solution to the auxiliary problem. It is clear that any feasible solution yields a profit that is at least as high as the profit of the efficient auction. Otherwise, the payoff floor constraint would be violated at $t = 0$. The following proposition shows that if positive reserve prices are feasible, that is, if the feasible set includes a solution with delayed trade for low types, then the seller can achieve a strictly higher revenue than in the efficient auction.

Proposition 4. *An efficient auction (T^E) is an optimal solution to the auxiliary problem if and only if it is the only feasible solution.*

To get an intuition for this result, compare the efficient auction in which all types trade at time zero, to an alternative feasible solution in which only the types in $(v_0^+, 1]$ trade at time zero, where $v_0^+ < 1$.[34] There are two effects that determine how the profits of these two solutions are ranked. First, in the alternative, the trade of low types is delayed, which creates an inefficiency. Second, the delay for the low types reduces information rents for higher types. We must argue that the total reduction in information rents exceeds the inefficiency, so that the ex-ante profit is higher under the alternative solution. We first consider the reduction in information rents only for the types in $[0, v_0^+]$. This is what matters for the continuation profit at time 0^+, that is, right after the initial trade. Feasibility implies that the reduction in information rents for the types in $[0, v_0^+]$ must already (weakly) exceed the revenue loss from inefficiency. Otherwise, the continuation profit at 0^+ would be smaller than the profit from an efficient auction given the posterior v_0^+, and thus the payoff floor constraint would be violated. If we now include the types in $(v_0^+, 1]$ in the comparison, we must add the reduction in information rents for these types but there is no additional inefficiency because these types trade at time zero in both solutions. Therefore, the total reduction in information rents is strictly higher than the inefficiency, and the ex-ante profit under the alternative is strictly higher than under the efficient auction.

Before proving Proposition 4, we first establish a lemma. We consider solutions where a strictly positive measure of types trade at the same time t so that $v_t > v_t^+$. In other words, there is an "atom" of types that trade at t. The following lemma shows that if the payoff floor constraint is satisfied right after the atom, then the payoff floor constraint at t (right

[34] In the proof of Proposition 4, we show that we can always construct a feasible solution with $0 < v_0^+ < 1$, if there exists any feasible solution that differs from the efficient auction.

before the atom) is strictly slack. Moreover, if we reduce the size of the atom by lowering v_t to $v \in (v_t^+, v_t)$ so that some types in the atom trade earlier than t, the payoff floor constraint at t remains strictly slack for all choices $v \in (v_t^+, v_t)$. This lemma is more general than needed for the proof of Proposition 4 which will be convenient later. The proof can be found in the Supplemental Material.

Lemma 6. *Let $T : [0,1] \to [0,1]$ be non-increasing (not necessarily feasible) and denote the corresponding cutoff sequence by v_t. Suppose there is an "atom" at $t \geq 0$, that is, $v_t > v_t^+$. If the payoff floor constraint is satisfied at t^+, that is*

$$\int_0^{v_t^+} e^{-r(T(x)-t)} \left(x - \frac{F(v_t^+) - F(x)}{f(x)} \right) dF^{(n)}(x) \geq \int_0^{v_t^+} \left(x - \frac{F(v_t^+) - F(x)}{f(x)} \right) dF^{(n)}(x), \tag{A.2}$$

then we have, for all $v \in (v_t^+, v_t]$,

$$\int_0^{v} e^{-r(T(x)-t)} \left(x - \frac{F(v) - F(x)}{f(x)} \right) dF^{(n)}(x) \geq \int_0^{v} \left(x - \frac{F(v) - F(x)}{f(x)} \right) dF^{(n)}(x). \tag{A.3}$$

In particular, the payoff floor constraint is satisfied at t. The inequality (A.3) is strict if $v_t^+ > 0$.

Proof of Proposition 4. The "if" part is trivial. For the "only if" part, suppose there is another feasible solution $\tilde{T}$ other than the efficient auction $T^E \equiv 0$. Let $\tilde{v}_t$ denote the cutoff path corresponding to $\tilde{T}$. Note first that the range of $\tilde{T}$ cannot be a singleton because this would imply that $\tilde{T}(v) = t$ for all $v \in [0,1]$ for some $t > 0$. Then the expected revenue would be given by

$$e^{-rt} \int_0^1 J(v) dF^{(n)}(v),$$

which is strictly lower than the revenue from an efficient auction at time 0. Therefore, the payoff floor constraint would be violated at $t = 0$, contradicting the feasibility of $\tilde{T}$.

Hence, there exists some time s with $0 < \tilde{v}_s^+ = \tilde{v}_s < 1$ such that $\tilde{T}(v) < s$ for all $v > \tilde{v}_s$, and $\tilde{T}(v) > s$ for all $v < \tilde{v}_s$. Then we can define a new feasible solution

$$\hat{T}(v) := \begin{cases} 0 & \text{if } v > \tilde{v}_s, \\ \tilde{T}(v) - s & \text{if } v \leq \tilde{v}_s, \end{cases}$$

with corresponding cutoff path $\hat{v}_t$. Solution $\hat{T}$ is feasible because $\tilde{T}$ satisfies the payoff floor constraint for all $t \geq s$. Moreover, we have $0 < \hat{v}_0^+ < 1$ because $\hat{v}_0^+ = \tilde{v}_s$. We can invoke

Lemma 6 by setting $t = 0$ and $v = v_0 = 1$ to obtain

$$\int_0^1 e^{-r\hat{T}(x)} J(x) dF^{(n)}(x) > \int_0^1 J(x) dF^{(n)}(x).$$

The left hand side of the above inequality is the revenue from $\hat{T}$, while the right hand side is the revenue from $T^E \equiv 0$. This completes the proof. □

Proposition 4 implies that in order to decide whether the efficient auction is optimal or not, it suffices to determine whether it is the unique feasible solution. This will be particularly useful, if we are able to construct solutions with non-zero trading times. We approach such a construction by considering the binding payoff floor constraint.

Lemma 7. *Let v_t be a sequence of cutoffs for which the payoff floor constraint is binding for all $t \in (a, b)$, where $0 \leq a < b \leq \infty$. Then v_t is twice continuously differentiable on (a, b) and satisfies the differential equation*

$$\frac{\ddot{v}_t}{\dot{v}_t} + g(v_t)\dot{v}_t + r = 0. \tag{A.4}$$

This lemma is a consequence of Lemmas 10 and 11 in Appendix B.6.1. The next lemma studies the solutions to the differential equation (A.4). In particular, we characterize precise conditions under which there exists a solution that is non-increasing and thus is feasible in the auxiliary problem. It turns out that a feasible solution exists if $n < \overline{N}(F)$ and does not exist if $n > \overline{N}(F)$.

Lemma 8. *(i) If $n > \overline{N}(F)$, there exists no non-increasing solution to (A.4) that satisfies $v_0^+ > 0$ and $\lim_{t\to\infty} v_t = 0$.*

(ii) If $n < \overline{N}(F)$, there exists a decreasing solution to (A.4) that satisfies $v_0^+ > 0$ and $\lim_{t\to\infty} v_t = 0$. If Assumption 1 holds, among all such solutions, the unique solution of (4.1) uniquely maximizes the seller's revenue for a given boundary value v_0^+.

Note that, when feasible solutions exist, they are not necessarily unique for a given boundary value v_0^+, because (A.4) is a second-order differential equation.[35] The second part of Lemma 8 identifies the optimal solution for a given boundary value. Lemma 8 follows from Lemmas 12–14 in Appendix B.6.2.

[35]The ODE (A.4) does not satisfy the Lipschitz condition at $v = 0$ because $g(v)$ may be unbounded. Therefore a boundary condition for $t \to \infty$ does not pin down a unique solution.

The last intermediate result is the following proposition which shows that the payoff floor constraint must be locally binding for an optimal solution if the monopoly profit is locally concave. It is clear from this proposition why the previous two lemmas are crucial for the analysis of optimal solutions to the auxiliary problem.

Proposition 5. *If $v(1-F(v))$ is locally concave on an interval $[0, v_a]$, then for every optimal solution, the payoff floor constraint binds for all t such that $v_t \in (0, v_a]$.*

Proposition 5 is a direct consequence of Lemmas 18 and 19 in Appendix B.6.3. In order to clarify the role of local concavity, we briefly outline the proof of Proposition 5. To show that the payoff floor constraint must bind at the optimal solution, we consider solutions for which the payoff floor constraint is slack for a time interval (a, b) and construct feasible variations. Roughly speaking, the variation we consider spreads out the trades that happen between a and b. For the high types in the interval $(v_b^+, v_a]$, we decrease the trading time, and for the low types we increase the trading time. Such a variation is always possible. If the monopoly profit $v(1 - F(v))$ is concave on the interval of valuations that trade between a and b, then we prove that such a variation is not only feasible but also improves the seller's ex-ante expected profit. If $v(1 - F(v))$ is convex, we have to construct a variation that concentrates the trading times of the types that trade between a and b, rather than spreading them out. Such a variation, however, is only feasible if the trade is not already concentrated on a single point in time. Therefore, with a non-concave monopoly profit, we cannot rule out that the payoff floor constraint is slack on some interval if there is an atom of trade at the end of the interval.[36]

In the proof of Theorem 2, we will use Proposition 5 on intervals of the form $(0, \varepsilon)$. In this case, the requirement of local concavity is satisfied for any distribution function without imposing Assumption 4 (see the discussion in Section 2). For Proposition 5 to have bite in this case, we show that a feasible solution cannot end with a single atom.

Lemma 9. *Let T be a feasible solution. Then for all $t > 0$ such that $v_t > 0$, there exists $w < v_t$ such that $T(v) > t$ for all $v \leq w$.*

Now we are ready to state the proofs of our main results.

[36] So far, we have not been able to rule out this possibility or to construct an example where a solution with this feature is optimal.

A.2 Proof of Theorem 1

Proof. From Proposition 1 we know that an optimal solution to the auxiliary problem exists and hence V is attained by an element in the feasible set. Lemma 4 implies that T^M is not in the feasible set of the auxiliary problem. Moreover, T^M is the only non-increasing trading time function that attains Π^M. Therefore $V < \Pi^M$. Proposition 3 then implies $\Pi^* \leq V < \Pi^M$. □

A.3 Proof of Theorems 2 and 3

A.3.1 Overview

The proofs of Theorems 2 and 3 each has two parts. The first characterizes the solution to the auxiliary problem. The second part shows that the value of the auxiliary problem is Π^* and that its optimal solution can be approximated by discrete time equilibria.

Theorem 2 assumes $n > \overline{N}(F)$. We use an indirect argument to show that in this case, the feasible set of the auxiliary problem only contains the efficient auction. Suppose by contradiction, that there exists another element T in the feasible set (we identify trading time functions that coincide almost everywhere). Proposition 4 implies that this solution yields strictly higher revenue than the efficient auction. T need not be optimal but Proposition 1 implies that an optimal solution to the auxiliary problem exists, which we call $\hat{T}$ with cutoffs denoted by $\hat{v}_t$. Remember that for any distribution, $v(1 - F(v))$ is locally concave for v sufficiently small. Therefore, Proposition 5 implies that the payoff floor constraint is locally binding for v sufficiently small. In other words, if t is large enough, so that $\hat{v}_t$ is sufficiently small, $\hat{v}_t$ must satisfy the ODE (A.4) and $\lim_{t\to\infty} \hat{v}_t = 0$.[37] This is a contradiction because Lemma 8 shows that for $n > \overline{N}(F)$ the ODE (A.4) does not admit a solution that satisfies $\lim_{t\to\infty} v_t = 0$. Therefore, we have shown that the feasible set of the auxiliary problem collapses to a singleton—the efficient auction—if $n > \overline{N}(F)$. In other words, $V = \Pi^E$. Note that this proof does not require Assumption 4 because local concavity around zero is a property of any distribution function with support [0,1].

For the second step in the proof of Theorem 2, note that $V = \Pi^E$, together with Proposition 3 and Lemma 3, implies that $\Pi^* = \Pi^E$ if $n > \overline{N}(F)$. Here we implicitly used equilibrium existence (Proposition 2.(i)), but do not require the uniform Coase conjecture in Proposition 2.(ii). Proposition 3 and Lemma 3 alone imply that equilibrium profits converge to Π^E.

[37] For this step, we need to ensure that the sequence $\hat{v}_t$ does not jump over the range of values where local concavity is guaranteed (see Lemma 9 in Appendix B.6.3).

Hence, the proof of Theorem 2 does not rely on Assumption 3.

Theorem 3 assumes $n < \overline{N}(F)$. In this case, Lemma 8 implies that the ODE (A.4) yields a feasible solution for the auxiliary problem. Taking $v_0^+ > 0$ we thus obtain a feasible solution that is different from the efficient auction. By Proposition 4, this solution must yield strictly higher revenue than the efficient auction. This establishes that the value of the auxiliary problem exceeds Π^E if $n < \overline{N}(F)$. For parts (ii) and (iii), Theorem 3 assumes global concavity (Assumptions 4). Under this assumption, Proposition 5 and Lemma 8 imply that the solution to the ODE (4.1) is an optimal solution (for an optimally chosen boundary condition v_0^+). By varying v_0^+ between 0 and the optimal value, we thus obtain a family of feasible solutions of the auxiliary problem that achieve any profit in $[\Pi^E, V]$.

For the second step in the proof of Theorem 3, we show that each solution in this family can be approximated by discrete time equilibria and thus establish sufficiency of the auxiliary problem (see Appendix D in the Supplemental Material). The approximation uses a discrete trading time $T^\Delta : [0,1] \to \{0, \Delta, 2\Delta, ...\}$, where $\Delta > 0$ is an arbitrarily chosen period length. T^Δ is constructed such that the payoff floor constraint is slack for all $t \in \{0, \Delta, 2\Delta, ...\}$. This approximation, together with (A.1), will be used to define the equilibrium price path for a game with given Δ. On the equilibrium path, buyers best respond to this price path. If the seller deviates from the equilibrium price path, the buyers use a continuation strategy given by a stationary equilibrium. Note that buyers can react to a deviation by the seller in the same period. Therefore, the response to a deviation is immediate and the seller cannot obtain profits in excess of the stationary equilibrium profit. The uniform Coase conjecture (Proposition 2.(ii)) thus implies that the profit after a deviation converges to the profit of the efficient auction. The equilibrium path, on the other hand is carefully constructed such that it yields a profit above the profit of stationary equilibria. As $\Delta \to \infty$, T^Δ is constructed such that it converges to the solution to the binding payoff floor constraint, but sufficiently slowly so that stationary equilibria can be used to provide incentives for the seller.

A.3.2 Proof of Theorem 2

Proof. Since $\phi > -1$ by Assumption 2, there exists a valuation $\bar{v} > 0$ such that for all $v \in [0, \bar{v}]$, $(f'(v)v)/f(v) > -2$ which implies that $v(1 - F(v))$ is concave on this interval. Lemma 9 shows that the optimal solution to the auxiliary problem does not end with an atom. Therefore, Proposition 5 implies that there exists a time $\bar{t}$ with $v_{\bar{t}} \leq \bar{v}$ after which the payoff floor must be binding for all t at the optimal solution. Lemma 8 shows that this is not possible if $n > \overline{N}(F)$. Proposition 4 and the existence of an optimal solution (Proposition

1) therefore imply that the efficient auction is the only element in the feasible set of the auxiliary problem if $n > \overline{N}(F)$. This shows $V = \Pi^E$. Proposition 3 and Lemma 3 then imply that $\Pi^* = V = \Pi^E = \Pi_*$. Proposition 2 shows the existence of stationary equilibria, and since $\Pi^* = \Pi^E$, there must exist a sequence of stationary equilibria for which the seller's profit converges to Π^E. □

A.3.3 Proof of Theorem 3

Proof. (i) Lemma 8 shows that, if $n < \overline{N}(F)$, then there exists a feasible solution to the auxiliary problem that differs from the efficient auction. This result, together with Proposition 4, implies that the efficient auction is not the optimal solution of the auxiliary problem if $n < \overline{N}(F)$. Again by Lemma 8, a profit $\tilde{\Pi} > \Pi^E$ can be achieved by the solution to the binding payoff floor constraint for some $v_0^+ \in (0, 1)$. By Proposition 6 in Appendix D in the Supplemental Material, there exists a sequence of equilibria $(p_m, b_m) \in \mathcal{E}(\Delta_m)$, for $\Delta_m \to 0$ as $m \to \infty$, such that $\lim_{m\to\infty} \Pi^{\Delta_m}(p_m, b_m) = \tilde{\Pi}$. This implies $\Pi^* > \Pi^E$.

(ii) By Proposition 5 and Assumption 4, the payoff floor constraint must be binding at the optimal solution to the auxiliary problem. By Lemma 8 the optimal solution must satisfy (4.1) and is unique up to the choice of v_0^+. If we choose v_0^+ optimally, we thus obtain the optimal solution to the auxiliary problem which achieves V. As in (i), Proposition 6 in Appendix D in the Supplemental Material implies that there exists a sequence of equilibria $(p_m, b_m) \in \mathcal{E}(\Delta_m)$, for $\Delta_m \to 0$ as $m \to \infty$, such that $\lim_{m\to\infty} \Pi^{\Delta_m}(p_m, b_m) = V$.

(iii) Let v_t^x be the sequence of cutoffs obtained from the ODE in (4.1) with boundary condition $v_0^+ = x \in [0, 1]$ and denote the value of the objective function of the auxiliary problem evaluated at v_t^x by $\Pi(x)$. The argument used in (ii) imply that for any choice $v_0^+ = x \in [0, 1]$, there exists a sequence of equilibria for which the equilibrium profits converge to $\Pi(x)$. This it remains to show that the range of $\Pi(x)$ is $[\Pi^E, \Pi^*]$. It is clear that $x = 0$ leads to $\Pi(x) = \Pi^E$ and from (ii) we know that there exists x^* such that $\Pi(x^*) = \Pi^*$. To complete the proof we show that $\Pi(x)$ is continuous. To see this, denote the trading time function corresponding to v_t^x by T^x. $\Pi(x)$ is obtained by substituting $T(v) = T^x(v)$ in the objective function of the auxiliary problem. Note that

$$T^x(v) = \begin{cases} 0, & \text{if } v \geq x, \\ T^1(v) - T^1(x), & \text{if } v \leq x. \end{cases}$$

Hence $T^x(v)$ is continuous in x for all $v > 0$ and therefore $e^{-rT^x(v)}$ is continuous in x for all

$v > 0$. Since $e^{-rT^x(v)}$ is bounded, $\Pi(x)$ is continuous in x, which completes the proof. □

A.3.4 Proof of Corollary 1

Proof. Since $f(0) \in (0, \infty)$, $\lim_{v\to 0} v f'(v)$ exists by Assumption 2. We first show that $\lim_{v\to 0} v f'(v) = 0$. Suppose by contradiction that $\lim_{v\to 0} v f'(v) = z \neq 0$. If $z > 0$, we must have $f'(v) \geq z/(2v)$ for a neighborhood $(0, \varepsilon)$, which implies that $f(\varepsilon) = f(0) + \int_0^\varepsilon f'(v)dv \geq f(0) + \int_0^\varepsilon (z/(2v))dv = \infty$ which contradicts the assumption of a finite density. If $z < 0$, we have $f'(v) \leq z/(2v)$ for a neighborhood $(0, \varepsilon)$, which implies that $f(\varepsilon) = f(0) + \int_0^\varepsilon f'(v)dv \leq f(0) + \int_0^\varepsilon (z/(2v))dv = -\infty$ which contradicts $f(v) > 0$. Since $f(0) > 0$ and $\lim_{v\to 0} v f'(v) = 0$ together imply $\phi = \lim_{v\to 0} \frac{vf'(v)}{f(v)} = 0$, we have $\overline{N}(F) := 1 + \frac{\sqrt{2+\phi}}{1+\phi} = 1 + \sqrt{2} \in (2, 3)$. □

A.3.5 Proof of Corollary 2

Proof. We use a Taylor expansion of $f(v)$ at zero to obtain

$$\phi = \lim_{v\to 0} \frac{f'(v)v}{f(v)} = \lim_{v\to 0} \frac{f'(v)v}{f'(0)v} = 1.$$

This implies $\overline{N}(F) = 1 + \sqrt{3}/2 < 2$. □

References

Ausubel, L. M., and R. J. Deneckere (1989): "Reputation in Bargaining and Durable Goods Monopoly," *Econometrica*, 57(3), 511–531.

Bergin, J., and W. B. MacLeod (1993): "Continuous Time Repeated Games," *International Economic Review*, 34(1), 21–27.

Bester, H., and R. Strausz (2000): "Imperfect Commitment and the Revelation Principle: The Multi-Agent Case," *Economics Letters*, 69, 165–171.

——— (2001): "Contracting with Imperfect Commitment and the Revelation Principle: The Single Agent Case," *Econometrica*, 69(4), 1007–1098.

Blume, A., and P. Heidhues (2004): "All Equilibria of the Vickrey Auction," *Journal of Economic Theory*, 114, 170–177.

Bulow, J., and P. Klemperer (1996): "Auctions Versus Negotiations," *American Economic Review*, 86, 180–194.

Bulow, J., and J. Roberts (1989): "The Simple Economics of Optimal Auctions," *The Journal of Political Economy*, 97(5), 1060–1090.

Chen, C.-H. (2012): "Name Your own Price at Priceline.com: Strategic Bidding and Lockout Periods," *Review of Economic Studies*, 79(4), 1341–1369.

Coase, R. H. (1972): "Durability and Monopoly," *Journal of Law and Economics*, 15(1), 143–149.

David, H. A., and H. N. Nagaraja (2003): *Order Statistics*. Wiley-Interscience, 3rd edn.

Fuchs, W., and A. Skrzypacz (2010): "Bargaining with Arrival of New Traders," *American Economic Review*, 100, 802–836.

Fudenberg, D., D. Levine, and J. Tirole (1985): "Infinite-Horizon Models of Bargaining with One-Sided Incomplete Information," in *Game-Theoretic Models of Bargaining*, ed. by A. Roth, chap. 5, pp. 73–98. Cambridge University Press.

Fudenberg, D., and J. Tirole (1991): *Game Theory*. MIT Press, Cambridge, Massachusetts.

Gul, F., H. Sonnenschein, and R. Wilson (1986): "Foundations of Dynamic Monopoly and the Coase Conjecture," *Journal of Economic Theory*, 39(1), 155–190.

Hart, O. D., and J. Tirole (1988): "Contract Renegotiation and Coasian Dynamics," *Review of Economic Studies*, 55(4), 509–540.

Hörner, J., and L. Samuelson (2011): "Managing Strategic Buyers," *Journal of Political Economy*, 119(3), 379–425.

Lee, J., and Q. Liu (2013): "Gambling Reputation: Repeated Bargaining with Outside Options," *Econometrica*, 81(4), 1601–1672.

McAdams, D., and M. Schwarz (2007): "Credible Sales Mechanisms and Intermediaries," *American Economic Review*, 97(1), 260–276.

McAfee, R. P., and J. McMillan (1987): "Auctions with a Stochastic Number of Bidders," *Journal of Economic Theory*, 43(1), 1–19.

McAfee, R. P., and D. Vincent (1997): "Sequentially Optimal Auctions," *Games and Economic Behavior*, 18, 246–276.

Milgrom, P. (1987): "Auction Theory," in *Advances in Economic Theory*, ed. by T. F. Bewley, no. 12 in Economic Society Monographs, pp. 1–32. Cambridge University Press.

Myerson, R. B. (1981): "Optimal Auction Design," *Mathematics of Operations Reseach*, 6, 58–63.

Riley, J. G., and W. F. Samuelson (1981): "Optimal Auctions," *American Economic Review*, 71(3), 381–392.

Skreta, V. (2006): "Sequentially Optimal Mechanisms," *Review of Economic Studies*, 73(4), 1085–1111.

——— (2016): "Optimal Auction Design under Non-Commitment," *Journal of Economic Theory*, 159, 854–890.

Strulovici, B. (2013): "Contract Renegotiation and the Coase Conjecture," unpublished manuscript, Northwestern University.

Vartiainen, H. (2013): "Auction Design without Commitment," *Journal of the European Economic Association*, 11, 316–342.

Wolitzky, A. (2010): "Dynamic monopoly with relational incentives," *Theoretical Economics*, 5, 479–518.