Calculation of elements of spin groups using method of averaging in Clifford's geometric algebra

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Real Clifford algebra, notations

Let us consider the real Clifford algebra $\mathcal{C}\!\ell_{p,q}$, $p+q=n$, with the identity element e and the generators e_a, $a=1,\ldots,n$, satisfying

$$e_a e_b + e_b e_a = 2\eta_{ab} e, \qquad \eta = ||\eta_{ab}|| = \mathrm{diag}(\underbrace{1,\ldots,1}_{p}, \underbrace{-1,\ldots,-1}_{q}). \tag{1}$$

We use notation with ordered multi-indices A for the basis elements of the Clifford algebra $\mathcal{C}\!\ell_{p,q}$:

$$e_A = e_{a_1\ldots a_k}, \qquad 1 \leq a_1 < \cdots < a_k \leq n. \tag{2}$$

We denote the length of multi-index A by $|A|$. In the particular case of the identity element e, we have multi-index of the length $|A|=0$.
We denote inverses of generators by

$$e^a := \eta^{ab} e_b = (e_a)^{-1}, \qquad a = 1,\ldots,n. \tag{3}$$

We call the subspace of $C\ell_{p,q}$ of Clifford algebra elements, which are linear combinations of basis elements with multi-indices of length $|A| = k$, the subspace of grade k and denote it by $C\ell^k_{p,q}$.
We denote even subspace (subalgebra) by $C\ell^{(0)}_{p,q}$ and odd subspace by $C\ell^{(1)}_{p,q}$.
We have

$$C\ell_{p,q} = \bigoplus_{k=0}^{n} C\ell^k_{p,q}, \qquad C\ell^{(j)}_{p,q} = \bigoplus_{k=j \mod 2} C\ell^k_{p,q}, \qquad j = 0, 1.$$

We denote grade involution (main involution) in Clifford algebra by

$$\widehat{U} := U|_{e_a \to -e_a}, \qquad U \in C\ell_{p,q} \tag{4}$$

and reversion (anti-involution) by

$$\widetilde{U} := U|_{e_{a_1 \ldots a_k} \to e_{a_k} \ldots e_{a_1}}, \qquad U \in C\ell_{p,q}. \tag{5}$$

Orthogonal groups

Let us consider the pseudo-orthogonal group

$$\mathrm{O}(p,q):=\{P\in\mathrm{Mat}(n,\mathbb{R}):P^{\mathrm{T}}\eta P=\eta\},\qquad p+q=n. \tag{6}$$

Let us use the following notation for the minors of the matrix $P=||p^a_b||$:

$$p^A_B=p^{a_1\ldots a_k}_{b_1\ldots b_k},\qquad a_1<\cdots<a_k,\qquad b_1<\cdots<b_k. \tag{7}$$

In the case of empty multi-indices A, B, the corresponding minor equals 1.

$$P\in\mathrm{O}(p,q)\Rightarrow \det P=\pm1,\quad |p^{1\ldots p}_{1\ldots p}|\geq 1,\quad |p^{p+1\ldots,n}_{p+1\ldots n}|\geq 1,\quad p^{1\ldots p}_{1\ldots p}=\frac{p^{p+1\ldots n}_{p+1\ldots n}}{\det P}.$$

The group $\mathrm{O}(p,q)$ has the following subgroups:

$$\mathrm{SO}(p,q):=\{P\in\mathrm{O}(p,q):\det P=1\}=\{P\in\mathrm{O}(p,q):p^{1\ldots p}_{1\ldots p}\geq1,p^{p+1\ldots n}_{p+1\ldots n}\geq1\},$$

$$\mathrm{O}_+(p,q):=\{P\in\mathrm{O}(p,q):p^{1\ldots p}_{1\ldots p}\geq1\},$$

$$\mathrm{O}_-(p,q):=\{P\in\mathrm{O}(p,q):p^{p+1\ldots n}_{p+1\ldots n}\geq1\},$$

$$\mathrm{SO}_+(p,q):=\{P\in\mathrm{SO}(p,q):p^{1\ldots p}_{1\ldots p}\geq1\}=\{P\in\mathrm{SO}(p,q):p^{p+1\ldots n}_{p+1\ldots n}\geq1\}.$$

The group $\mathrm{O}(p, q)$ has four components in the case $p \neq 0$, $q \neq 0$:

$$\mathrm{O}(p,q) = \mathrm{SO}_+(p,q) \sqcup \mathrm{O}_+(p,q)' \sqcup \mathrm{O}_-(p,q)' \sqcup \mathrm{SO}(p,q)'$$

$$\mathrm{O}_+(p,q) = \mathrm{SO}_+(p,q) \sqcup \mathrm{O}_+(p,q)', \quad \mathrm{O}_-(p,q) = \mathrm{SO}_+(p,q) \sqcup \mathrm{O}_-(p,q)'$$

$$\mathrm{SO}(p,q) = \mathrm{SO}_+(p,q) \sqcup \mathrm{SO}(p,q)'.$$

Example: In the case $(p, q) = (1, 3)$, we have Lorentz group $\mathrm{O}(1, 3)$, special (or proper) Lorentz group $\mathrm{SO}(1, 3)$, orthochronous Lorentz group $\mathrm{O}_+(1, 3)$, orthochorous (or parity preserving) Lorentz group $\mathrm{O}_-(1, 3)$, proper orthochronous Lorentz group $\mathrm{SO}_+(1, 3)$.

In the cases $p = 0$ or $q = 0$, we have orthogonal group

$$\mathrm{O}(n) := \mathrm{O}(n, 0) \cong \mathrm{O}(0, n)$$

and special orthogonal group

$$\mathrm{SO}(n) := \mathrm{SO}(n, 0) \cong \mathrm{SO}(0, n).$$

The group $\mathrm{O}(n)$ has two connected components:

$$\mathrm{O}(n) = \mathrm{SO}(n) \sqcup \mathrm{SO}(n)'.$$

Let us consider the set

$$\beta_a := p_a^b e_b, \tag{8}$$

Then

$$\beta_a\beta_b + \beta_b\beta_a = 2\eta_{ab}e \quad \Leftrightarrow \quad P = ||p_a^b|| \in \mathrm{O}(p,q). \tag{9}$$

For (8), (9) let us consider the elements

$$\beta_A = \beta_{a_1\ldots a_k} := \beta_{a_1}\cdots\beta_{a_k}, \qquad 1 \leq a_1 < \cdots < a_k \leq n. \tag{10}$$

Lemma

For (8) we have

$$\beta_{a_1\ldots a_k} = p_{a_1\ldots a_k}^{b_1\ldots b_k} e_{b_1\ldots b_k}, \qquad (\beta_A = p_A^B e_B) \tag{11}$$

where $p_{a_1\ldots a_k}^{b_1\ldots b_k}$ are minors of the matrix $P = ||p_a^b|| \in \mathrm{O}(p,q)$ and $p_\varnothing^\varnothing := 1$.

(We use Einstein summation convection for ordered multi-indices too.)
Note that as particular case of (11) we have

$$\beta_{1\ldots n} = \det(P)e_{1\ldots n}, \qquad \det P = \pm 1.$$

Spin groups

We denote by $M^{\times}$ the subset of invertible elements of the set M.
The following group is called Lipschitz group

$$\Gamma^{\pm}_{p,q}=\{S\in C\!\ell^{(0)\times}_{p,q}\cup C\!\ell^{(1)\times}_{p,q}: S^{-1}C\!\ell^{1}_{p,q}S\subset C\!\ell^{1}_{p,q}\}=\{v_1\cdots v_k : v_1,\ldots,v_k\in C\!\ell^{1\times}_{p,q}\}.$$

Let us consider the following subgroup of Lipschitz group

$$\Gamma^{+}_{p,q}:=\{S\in C\!\ell^{(0)\times}_{p,q}: S^{-1}C\!\ell^{1}_{p,q}S\subset C\!\ell^{1}_{p,q}\}=\{v_1\cdots v_{2k} : v_1,\ldots,v_{2k}\in C\!\ell^{1\times}_{p,q}\}\subset\Gamma^{\pm}_{p,q}.$$

The following groups are called spin groups:

$$\begin{aligned}
\mathrm{Pin}(p,q) &:= \{S\in\Gamma^{\pm}_{p,q} : \widetilde{S}S=\pm e\}=\{S\in\Gamma^{\pm}_{p,q} : \widehat{\widetilde{S}}S=\pm e\},\\
\mathrm{Pin}_{+}(p,q) &:= \{S\in\Gamma^{\pm}_{p,q} : \widehat{\widetilde{S}}S=+e\},\\
\mathrm{Pin}_{-}(p,q) &:= \{S\in\Gamma^{\pm}_{p,q} : \widetilde{S}S=+e\},\\
\mathrm{Spin}(p,q) &:= \{S\in\Gamma^{+}_{p,q} : \widetilde{S}S=\pm e\}=\{S\in\Gamma^{+}_{p,q} : \widehat{\widetilde{S}}S=\pm e\},\\
\mathrm{Spin}_{+}(p,q) &:= \{S\in\Gamma^{+}_{p,q} : \widetilde{S}S=+e\}=\{S\in\Gamma^{+}_{p,q} : \widehat{\widetilde{S}}S=+e\}.
\end{aligned}\tag{12}$$

In the case $p \neq 0$ and $q \neq 0$, we have

$$\mathrm{Pin}(p,q) = \mathrm{Spin}_+(p,q) \sqcup \mathrm{Pin}_+(p,q)' \sqcup \mathrm{Pin}_-(p,q)' \sqcup \mathrm{Spin}(p,q)'$$

$$\mathrm{Pin}_+(p,q) = \mathrm{Spin}_+(p,q) \sqcup \mathrm{Pin}_+(p,q)', \quad \mathrm{Pin}_-(p,q) = \mathrm{Spin}_+(p,q) \sqcup \mathrm{Pin}_-(p,q)'$$

$$\mathrm{Spin}(p,q) = \mathrm{Spin}_+(p,q) \sqcup \mathrm{Spin}(p,q)'.$$

In Euclidean cases, we have

$$\mathrm{Pin}(n) := \mathrm{Pin}(n,0) = \mathrm{Pin}_-(n,0) \ncong \mathrm{Pin}(0,n) := \mathrm{Pin}(0,n) = \mathrm{Pin}_+(0,n),$$

$$\mathrm{Spin}(n) := \mathrm{Spin}(n,0) = \mathrm{Pin}_+(n,0) = \mathrm{Spin}_+(n,0) \cong$$

$$\cong \mathrm{Spin}(0,n) = \mathrm{Pin}_-(0,n) = \mathrm{Spin}_+(0,n).$$

Details about all 5 spin groups and 5 orthogonal groups:

- I. M. Benn, R. W. Tucker, *An introduction to Spinors and Geometry with Applications in Physics*, Bristol, 1987.
- D. Shirokov, *Clifford algebras and their applications to Lie groups and spinors*, Lectures, Sofia, Bulgaria (2018), 11–53, arXiv:1709.06608

Let us consider the twisted adjoint representation

$$\psi : C\ell^{\times}_{p,q} \to \mathrm{End}\, C\ell_{p,q}, \quad S \to \psi_S, \quad \psi_S(U) = S^{-1} U \widehat{S},\ U \in C\ell_{p,q}.$$

The following homomorphisms are surjective with the kernel $\{\pm 1\}$:

$$\begin{aligned}
\psi &: \mathrm{Pin}(p,q) \to \mathrm{O}(p,q)\\
\psi &: \mathrm{Spin}(p,q) \to \mathrm{SO}(p,q)\\
\psi &: \mathrm{Spin}_+(p,q) \to \mathrm{SO}_+(p,q)\\
\psi &: \mathrm{Pin}_+(p,q) \to \mathrm{O}_+(p,q)\\
\psi &: \mathrm{Pin}_-(p,q) \to \mathrm{O}_-(p,q).
\end{aligned}$$

It means that

$$\text{for any } P = ||p^a_b|| \in \mathrm{O}(p,q) \text{ there exists } \pm S \in \mathrm{Pin}(p,q) : S^{-1} e_a \widehat{S} = p^b_a e_b \quad (13)$$

and for the other groups similarly. We can say that these spin groups are two-sheeted coverings of the corresponding orthogonal groups.

Hestenes method, the case $(p, q) = (1, 3)$

For each element $P = ||p^a_b|| \in \mathrm{SO}_+(1,3)$ there exist two elements $\pm S \in \mathrm{Spin}_+(1,3)$ such that

$$S^{-1} e_a S = p^b_a e_b. \tag{14}$$

The elements $\pm S$ can be found in the following way

$$S = \pm \frac{\widetilde{L}}{\sqrt{\widetilde{L}L}}, \qquad L := p^b_a e_b e^a. \tag{15}$$

Note that this method works in the case of dimension $n = 4$ and for the matrices $P = ||p^a_b|| \in \mathrm{SO}_+(1,3)$ with additional condition

$$\widetilde{L}L \neq 0 \qquad (\Leftrightarrow L \neq 0). \tag{16}$$

Note that the condition (16) for the orthogonal matrix is equivalent to the condition

$$\pi_0(S) \neq 0 \qquad \text{or} \qquad \pi_4(S) \neq 0 \tag{17}$$

for the corresponding element of the spin group $S \in \mathrm{Spin}_+(1,3)$. We denote projection operator onto the subspace of grade k by π_k, $k = 1, \ldots, n$.

Plan of the proof:

$$S^{-1}e_aS = p_a^b e_b \quad | \cdot e^a, \tag{18}$$

$$S^{-1}(e_aSe^a) = p_a^b e_b e^a := L, \tag{19}$$

$$e_aS_ke^a = (-1)^k(n-2k)S_k, \quad S_k \in C\ell^k_{1,3}, \tag{20}$$

$$S \in \mathrm{Spin}_+(1,3) \Rightarrow S = S_0 + S_2 + S_4, \tag{21}$$

$$S^{-1}4(S_0 - S_4) = L, \tag{22}$$

$$4(S_0 - S_4)\widetilde{S^{-1}} = \widetilde{L}, \tag{23}$$

$$4(S_0 - S_4)(S\widetilde{S})^{-1}4(S_0 - S_4) = \widetilde{L}L, \qquad S\widetilde{S} = e, \tag{24}$$

$$(4(S_0 - S_4))^2 = \widetilde{L}L \in C\ell^0_{1,3} \oplus C\ell^4_{1,3} \cong \mathbb{C}, \tag{25}$$

$$4(S_0 - S_4) = \sqrt{\widetilde{L}L}, \tag{26}$$

$$S = \pm\frac{\widetilde{L}}{\sqrt{\widetilde{L}L}}. \tag{27}$$

Hestenes D., *Space-Time Algebra*, Gordon and Breach, New York 1966.

Lounesto P., *Clifford Algebras and Spinors*, Cambridge 2001.

Other methods

Using exponents and exterior exponents:

Hestenes D., *Space-Time Algebra*, Gordon and Breach, New York 1966.

Hestenes D., Sobczyk G., *Clifford Algebra to Geometric Calculus*, Reidel Publishing Company, Dordrecht Holland, 1984.

Lounesto P., *Clifford Algebras and Spinors*, Cambridge Univ. Press, Cambridge 2001.

Doran C. and Lasenby A., *Geometric Algebra for Physicists*, Cambridge University Press, Cambridge 2003.

Marchuk N., *Parametrisations of elements of spinor and orthogonal groups using exterior exponents*, AACA, 21:3 (2011), 583–590, arXiv:0912.5349

Using Pauli's theorem (algorithm, not explicit formulas):

Shirokov D., *Calculations of Elements of Spin Groups Using Generalized Pauli's Theorem*, Advances in Applied Clifford Algebras **25**:1 (2015) 227–244, arXiv:1409.2449

Some ideas of the method of averaging:

Reynolds operator acting on a Clifford algebra element:

$$R_G(U) := \frac{1}{|G|}\sum_{g\in G} g^{-1}Ug, \qquad U\in C\ell_{p,q}, \tag{28}$$

where $|G|$ is the number of elements in finite subgroup $G\subset C\ell^{\times}_{p,q}$. We can take Salingaros group $G=\{\pm e_A\}$ and obtain operator

$$\frac{1}{2^n}e_A U e^A = \pi_{\mathrm{Cen}}(U) = \begin{cases} \pi_0(U), & \text{if } n=0 \mod 2;\\ \pi_0(U)+\pi_n(U), & \text{if } n=1 \mod 2.\end{cases} \tag{29}$$

$$\sum_{|A|=j \mod 2}(e_A S e^A), \quad j=0,1, \qquad \sum_{|A|=k \mod 4}(e_A S e^A), \quad k=0,1,2,3,$$

$$\sum_{|A|=m}(e_A S e^A), \quad m=1,2,\ldots,n.$$

Shirokov D., *Method of Averaging in Clifford Algebras*, Advances in Applied Clifford Algebras, **27**:1 (2017) 149–163.

Shirokov D., *Contractions on Ranks and Quaternion Types in Clifford Algebras*, Vestn. Samar. Gos. Tekhn. Univ., **19**:1 (2015) 117–135.

Plan of the new proof:

$$S^{-1}e_a\widehat{S} = p_a^b e_b, \tag{30}$$
$$S^{-1}e_a S = \det(P)p_a^b e_b, \tag{31}$$
$$S^{-1}e_A S = \sum_A (\det P)^{|A|} p_A^B e_B \quad | \cdot e^A, \tag{32}$$
$$S^{-1}(e_A S e^A) = \sum_A (\det P)^{|A|} p_A^B e_B e^A := M, \tag{33}$$
$$S^{-1}2^n \pi_{\mathrm{Cen}}(S) = M, \tag{34}$$

$\ldots$ different cases which depend on $n \mod 4$ and $p-q \mod 4$:

$$\widetilde{\pi_{\mathrm{Cen}}(S)}\widetilde{S^{-1}} = \widetilde{M}, \quad \widetilde{\pi_{\mathrm{Cen}}(S)}(S\widetilde{S})^{-1}\pi_{\mathrm{Cen}}(S) = \widetilde{M}M,\ S\widetilde{S} = \pm e := \alpha,$$

or

$$\widetilde{\widehat{\pi_{\mathrm{Cen}}(S)}}\widetilde{\widehat{S^{-1}}} = \widetilde{\widehat{M}}, \quad \widetilde{\widehat{\pi_{\mathrm{Cen}}(S)}}(S\widetilde{\widehat{S}})^{-1}\pi_{\mathrm{Cen}}(S) = \widetilde{\widehat{M}}M,\ S\widetilde{\widehat{S}} = \pm e := \alpha,$$

$\ldots$

Theorem

Let us consider the real Clifford algebra $C\ell_{p,q}$*, with even* $n=p+q$*. Let* $P\in\mathrm{SO}(p,q)$ *be an orthogonal matrix such that*

$$M:=\sum_{A,B}p^B_A e_B e^A\neq 0 \qquad (\Leftrightarrow \pi_{\mathrm{Cen}}(S)\neq 0). \tag{35}$$

We can find the elements $\pm S\in\mathrm{Spin}(p,q)$ *that correspond to* $P=||p^b_a||\in\mathrm{SO}(p,q)$ *as two-sheeted covering* $S^{-1}e_a S=p^b_a e_b$ *in the following way:*

$$S=\pm\frac{\widetilde{M}}{\sqrt{\alpha\,\widetilde{M}M}}, \qquad \text{where } \widetilde{M}M\in\mathrm{Cen}(C\ell_{p,q})=C\ell^0_{p,q}\cong\mathbb{R}, \tag{36}$$

and the sign

$$\alpha:=\mathrm{sign}(p^{1\ldots p}_{1\ldots p})e=\mathrm{sign}(p^{p+1\ldots n}_{p+1\ldots n})e=\widetilde{S}S=\pm e \tag{37}$$

depends on the component of the orthogonal group $\mathrm{SO}(p,q)$ *(or the corresponding component of the group* $\mathrm{Spin}(p,q)$*).*

Theorem

Let us consider the real Clifford algebra $C\ell_{p,q}$ with odd $n=p+q$. Let $P\in\mathrm{O}(p,q)$ be an orthogonal matrix such that

$$M:=\sum_{A,B}(\det P)^{|A|}p^B_A e_B e^A \neq 0 \qquad (\Leftrightarrow \pi_{\mathrm{Cen}}(S)\neq 0). \tag{38}$$

We can find the elements $\pm S\in\mathrm{Pin}(p,q)$ that correspond to $P=||p^b_a||\in\mathrm{O}(p,q)$ as two-sheeted covering $S^{-1}e_a\widehat{S}=p^b_a e_b$ in the following way:

$$S=\pm\frac{\widetilde{M}}{\sqrt{\alpha\,\widetilde{M}M}}, \tag{39}$$

where

$$\widetilde{M}M\in C\ell^0_{p,q}\subset\mathrm{Cen}(C\ell_{p,q})\cong\begin{cases}\mathbb{R}\oplus\mathbb{R}, & \text{if } p-q=1 \mod 4;\\ \mathbb{C}, & \text{if } p-q=3 \mod 4,\end{cases} \tag{40}$$

$$\alpha:=\begin{cases}\mathrm{sign}(p^{p+1\ldots n}_{p+1\ldots n})e=\widetilde{S}S=\pm e, & \text{if } n=1 \mod 4;\\ \mathrm{sign}(p^{1\ldots p}_{1\ldots p})e=\widehat{\widetilde{S}}S=\pm e, & \text{if } n=3 \mod 4.\end{cases} \tag{41}$$

Example: $n = 3$.

$$P \in \mathrm{SO}(3): \quad \det P = 1, \qquad S \in \mathrm{Spin}(3): \quad \alpha = \widehat{\widetilde{S}} S = \widetilde{S} S = +e,$$

$$M = \sum_{A,B} (\det P)^{|A|} p_A^B e_B e^A = e + p_a^b e_b e^a + p_{a_1 a_2}^{b_1 b_2} e_{b_1 b_2} e^{a_1 a_2} + e_{123} e^{123},$$

$$M = 2(e + p_a^b e_b e^a) = 2(e + \beta_a e^a), \qquad S = \pm \frac{\widetilde{M}}{\sqrt{\widetilde{M} M}}. \tag{42}$$

Doran C. and Lasenby A., *Geometric Algebra for Physicists*, Cambridge University Press, Cambridge 2003.

Generalization to the case of arbitrary $n = p + q$ for the rotor $S \in \mathrm{Spin}_+(p, q)$:

$$\widetilde{S} e_a S = \beta_a = p_a^b e_b, \quad S = \pm \frac{\widetilde{M}}{\sqrt{\widetilde{M} M}}, \quad M = \beta_A e^A = p_A^B e_B e^A. \tag{43}$$

Thank you for your attention!