The new KS method for a structural break detection in GARCH(1,1) models

Dmitriy Borzykh, Artem Yazykov

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Preface

The presentation is based on the results of our forthcoming article "The new KS method for a structural break detection in GARCH(1,1) models" in Applied Econometrics.

Model

Let $\tau \in \{1, \ldots, T\}$ be a possible single structural break moment which separates time series $Y = (Y_t)_{t=1}^{T}$ in two homogenous parts:

$$\begin{aligned} Y_t &= \varepsilon_t, \quad \varepsilon_t = \sigma_t \cdot \xi_t, \quad \sigma_t^2 = \omega_1 + \delta_1 \cdot \sigma_{t-1}^2 + \gamma_1 \cdot \varepsilon_{t-1}^2, \quad t \in [1;\, \tau - 1], \\ Y_t &= \varepsilon_t, \quad \varepsilon_t = \sigma_t \cdot \xi_t, \quad \sigma_t^2 = \omega_2 + \delta_2 \cdot \sigma_{t-1}^2 + \gamma_2 \cdot \varepsilon_{t-1}^2, \quad t \in [\tau;\, T], \end{aligned}$$

where $\theta_j := (\omega_j, \delta_j, \gamma_j)$, $j = 1, 2$, are unknown model parameters belonging to the set $\Theta := \big\{(\omega, \delta, \gamma)\colon\ \omega > 0,\, \delta \geq 0,\, \gamma \geq 0,\, \delta + \gamma < 1\big\}$,

and $(\xi_t)_{t=-\infty}^{+\infty}$ is a sequence of independent standard normal random variables.

KL method (Kokoszka, Leipus, 1999)

Consider the following statistics

$$\mathrm{KL}(k) := \frac{1}{\sqrt{T}}\left(\sum_{t=1}^{k} Y_t^2 - \frac{k}{T}\sum_{t=1}^{T} Y_t^2\right), \quad k \in \{1, \ldots, T\}.$$

The moment suspicious for structural break is determined by

$$\widehat{\tau}_{\mathrm{KL}} := \min\left\{k\colon\ \left|\mathrm{KL}(k)\right| = \max_{j\in\{1,\ldots,T\}}\left|\mathrm{KL}(j)\right|\right\}.$$

Put $r = \lfloor \sqrt{T} \rfloor$, where $\lfloor \cdot \rfloor$ stands for rounding down.

Denote $\hat{v}^2_{r,\,T} := \sum_{|j| \le r} w_j \hat{c}_j$, where $w_j := 1 - \frac{|j|}{r+1}$, and

$$\hat{c}_j := \frac{1}{T} \sum_{i=1}^{T-|j|} \left(Y_i^2 - \overline{Y^2} \right) \left(Y_{i+|j|}^2 - \overline{Y^2} \right).$$

Structural break criterion: if

$$\frac{\left| \mathrm{KL}(\widehat{\tau}_{\mathrm{KL}}) \right|}{\hat{v}_{r,\,T}} \ge q_{0.99},$$

than at significance level of 1% the moment $\widehat{\tau}_{\mathrm{KL}}$ is considered to be **the moment of structural break**, where $q_{0.99} = 1.628$ is 0.99 quantile of the Brownian bridge absolute value supremum $\sup_{u \in [0;\,1]} \left| B^0(u) \right|$.

IT method (Inclán, Tiao, 1994)

Consider the following statistics

$$\mathrm{IT}(k) := \frac{\sum_{t=1}^{k} \hat{\xi}_t^2}{\sum_{t=1}^{T} \hat{\xi}_t^2} - \frac{k}{T}, \quad k \in \{1, \ldots, T\},$$

where $\hat{\xi}_t = \hat{\varepsilon}_t / \hat{\sigma}_t$ are standardized residuals of GARCH process.
The moment suspicious for structural break is determined by

$$\widehat{\tau}_{\mathrm{IT}} := \min\Big\{k\colon\ \big|\mathrm{IT}(k)\big| = \max_{j\in\{1,\ldots,T\}} \big|\mathrm{IT}(j)\big|\Big\}.$$

Structural break criterion: if

$$\sqrt{T/2}\,\big|\mathrm{IT}(\widehat{\tau}_{\mathrm{IT}})\big| \geq q_{0.99},$$

than at significance level of 1% the moment $\widehat{\tau}_{\mathrm{IT}}$ is considered to be **the moment of structural break**, where $q_{0.99} = 1.628$.

LTM method (Lee, Tokutsu, Maekawa, 2004)

Consider the following statistics

$$\mathrm{LTM}(k) := \frac{1}{\sqrt{T}\hat{\eta}}\left|\sum_{t=1}^{k}\hat{\xi}_t^2 - \frac{k}{T}\sum_{t=1}^{T}\hat{\xi}_t^2\right|, \quad k \in \{1, \ldots, T\},$$

where $\hat{\eta}^2 := \frac{1}{T}\sum_{t=1}^{T}\hat{\xi}_t^4 - \left(\frac{1}{T}\sum_{t=1}^{T}\hat{\xi}_t^2\right)^2$ and $\hat{\xi}_t = \hat{\varepsilon}_t/\hat{\sigma}_t$ are standardized residuals of GARCH process.

The moment suspicious for structural break is determined by

$$\widehat{\tau}_{\mathrm{LTM}} := \min\left\{k : \ \mathrm{LTM}(k) = \max_{j\in\{1,\ldots,T\}} \mathrm{LTM}(j)\right\}.$$

Structural break criterion: if

$$\mathrm{LTM}(\widehat{\tau}_{\mathrm{LTM}}) \geq q_{0.99},$$

than at the significance level of 1% the moment $\widehat{\tau}_{\mathrm{LTM}}$ is considered to be **the moment of structural break**, where $q_{0.99} = 1.628$.

KS algorithm

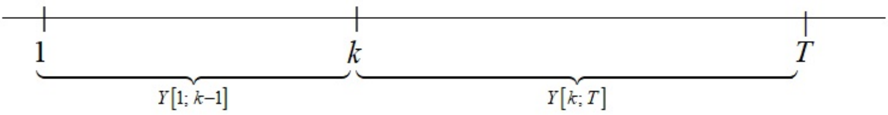
- Note at once that the proposed KS method is exclusively heuristic and does not have rigorous mathematical justification.
- In the situation with the GARCH process observations are not independent, as required in the Kolmogorov–Smirnov theorem.
- Therefore, strictly speaking, the Kolmogorov–Smirnov test is not applicable in this case.
- Despite this, numerical experiments show that the KS method has good statistical properties — sufficiently low probabilities of type I error and high power of detection of structural breaks in GARCH(1,1) models.

KS algorithm

- Let $\widehat{F}_Y(x)$ and $\widehat{F}_Z(x)$ be the sample distribution functions, constructed from samples Y and Z respectively.
- Let $\mathrm{dist}(Y, Z) := \sup_{x \in \mathbb{R}} |\widehat{F}_Y(x) - \widehat{F}_Z(x)|$ be uniform distance between these distribution functions.
- Denote $Y[s; t] := [Y_s, \ldots, Y_t]$, where $Y = [Y_1, \ldots, Y_T]$ and $1 \leq s \leq t \leq T$.

KS algorithm

- First of all, we explain the idea of the proposed method.
- Let us fix the parameters of our method — numbers $\Delta_1, \Delta_2 \in \mathbb{Z}_+$.
- Let us fix an arbitrary moment of time $k \in [\Delta_1;\ T - \Delta_1]$ and consider two subsamples $Y[1;\ k-1]$ and $Y[k;\ T]$, which are located in time "to the left" and "to the right" from the moment k, respectively. See figure below.



KS algorithm

- Let us divide the "left" sample $Y[1;\ k-1]$ into two subsamples $Y[1;\ \lfloor k/2 \rfloor]$ and $Y[\lfloor k/2 \rfloor + 1;\ k-1 \rfloor]$ of approximately the same volume.
- We calculate the distance between the sample distribution functions of these subsamples:

$$\mathfrak{D}_L(k) := \mathrm{dist}\Big(Y[1;\ \lfloor k/2 \rfloor],\ Y[\lfloor k/2 \rfloor + 1;\ k-1 \rfloor]\Big).$$

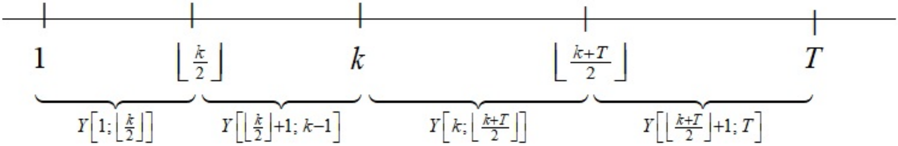
KS algorithm

- Similarly, we divide the "right" sample $Y[k;\ T]$ into two subsamples $Y[k;\ \lfloor (k+T)/2 \rfloor]$ and $Y[\lfloor (k+T)/2 \rfloor + 1;\ T]$ of approximately the same volume.
- We calculate the distance between the sample distribution functions of these subsamples:

$$\mathfrak{D}_R(k) := \mathrm{dist}\Big(Y[k;\ \lfloor (k+T)/2 \rfloor],\ Y[\lfloor (k+T)/2 \rfloor + 1;\ T] \Big).$$

KS algorithm

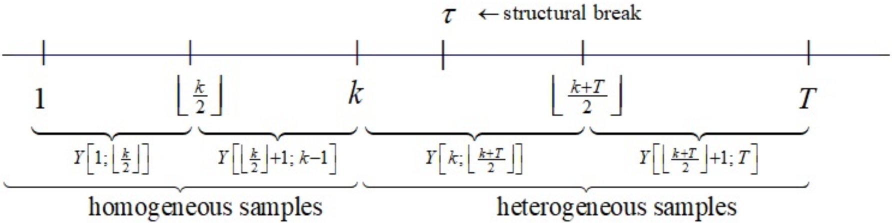
These steps are illustrated in the following figure.



KS algorithm

The idea of the method is the following remark. The moment of time k coincides with true moment of structural break τ if and only if

1. both samples $Y[1;\ \lfloor k/2 \rfloor]$ and $Y[\lfloor k/2 \rfloor + 1;\ k-1]$ are "homogeneous" and
2. both samples $Y[k;\ \lfloor (k+T)/2 \rfloor]$ and $Y[\lfloor (k+T)/2 \rfloor + 1;\ T]$ are "homogeneous".



KS algorithm

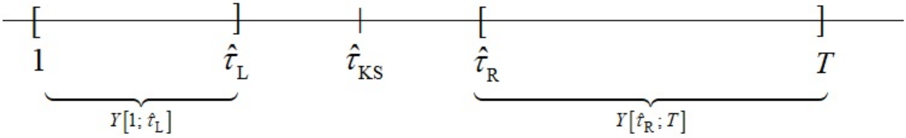
STEP 1 (ESTIMATION). We define a **moment suspicious for structural break** by formula

$$\widehat{\tau}_{\mathrm{KS}} \in \operatorname{argmin}_{k\in[\Delta_1;\, T-\Delta_1]}\big(\mathfrak{D}_L(k)+\mathfrak{D}_R(k)\big).$$

STEP 2 (VALIDATION). At a given significance level we apply Kolmogorov–Smirnov test to the samples $Y[1;\,\widehat{\tau}_L]$ and $Y[\widehat{\tau}_R;\ T]$, where

$$\widehat{\tau}_L := \max\{\widehat{\tau}_{\mathrm{KS}}-\Delta_2,\ \Delta_1\},\quad \widehat{\tau}_R := \min\{\widehat{\tau}_{\mathrm{KS}}+\Delta_2,\ T-\Delta_1\}.$$

If in this test null hypothesis is rejected, we consider the point $\widehat{\tau}_{\mathrm{KS}}$ to be a **structural break**. Otherwise, we believe that there is **no structural break** at this point.



Numeric experiments

- To make the experiments closer to real conditions, we generate GARCH processes with coefficients estimated on 26 Russian stock time series (see Table 1).
- The data were taken from FINAM's website for the period from 1 January 2011 to 31 December 2013.
- We conducted four numerical experiments, each of which consisted of 26 calculations.
- Each calculation consisted of 5 000 simulations.

Table 1. The list of analyzed stocks

Ticker	Name of Stock
AFKS	ПАО АФК «Система», ао
AFLT	ПАО «Аэрофлот», ао
ALRS	АК «АЛРОСА» (ПАО), ао
CHMF	ПАО «Северсталь», ао
FEES	ПАО «ФСК ЕЭС», ао
GMKN	ПАО «ГМК Норильский никель», ао
HYDR	ПАО «РусГидро», ао
IRAO	ПАО «Интер РАО», ао
LKOH	ПАО «ЛУКОЙЛ», ао
MAGN	ПАО «ММК», ао
MGNT	ПАО «Магнит», ао
MTSS	ПАО «МТС», ао
NLMK	ПАО «НЛМК», ао
NVTK	ПАО «НОВАТЭК», ао

Numeric experiments

Table 1 (continued). The list of analyzed stocks

Ticker	Name of Stock
PIKK	ПАО «Группа Компаний ПИК», ао
PLZL	ПАО «Полюс», ао
ROSN	ПАО «НК Роснефть», ао
RTKM	ПАО «Ростелеком», ао
SBER	ПАО Сбербанк, ао
SNGS	ОАО «Сургутнефтегаз», ао
SNGSP	ОАО «Сургутнефтегаз», ап
TATN	ПАО «Татнефть» им. В.Д. Шашина, ао
TATNP	ПАО «Татнефть» им. В.Д. Шашина, ап
TRMK	ПАО «ТМК», АО
TRNFP	ПАО «Транснефть», ап
VTBR	Банк ВТБ (ПАО), ао

Numerical experiments

- Significance level 1%.
- $\Delta_1 = 4,\ \Delta_2 = 400,\ T = 2\,000.$

Table 2. Average probabilities of type I error

	$\overline{\alpha}_{\mathrm{KS}}$	$\overline{\alpha}_{\mathrm{KL}}$	$\overline{\alpha}_{\mathrm{IT}}$	$\overline{\alpha}_{\mathrm{LTM}}$
experiment 1: no structural break	0.049	0.074	0.003	0.003

Table 3. Average powers ($\tau = 1\,001$)

jump type	$\overline{W}_{\mathrm{KS}}$	$\overline{W}_{\mathrm{KL}}$	$\overline{W}_{\mathrm{IT}}$	$\overline{W}_{\mathrm{LTM}}$
experiment 2: increase ω in 5 times	0.99	0.97	0.30	0.22
experiment 3: decrease δ by 0.1	0.84	0.87	0.42	0.38
experiment 4: decrease γ by 0.04	0.62	0.75	0.28	0.25

Summary

- The KL method has the highest power on average to detect structural breaks among the other methods.
- Our KS method has a slightly lower power while IT and LTM methods are dramatically less powerful.
- However, our KS method demonstrates lower probability of type I error on average.
- As a result, we suggest that our method is highly competitive and may be placed somewhere in between the KL method which has high power and high probability of type I error, and IT and LTM methods which have low power and also low probability of type I error.

Thank you for your attention!