Confidence Estimation of Transition Matrices Given Limited Data Sample

Viktor Lapshin,
Anton Markov

NRU Higher School of Economics

Agenda

- Why study the topic?
- Introduction to Probability of Default (PD) Modelling using Transition Matrices
- Experiment 1.
  Adequacy Check of Confidence Estimation Methods
- Experiment 2.
  Data Requirements for Fixed Conf. Intervals (CI) Widths
- Practical Example
  Confidence Intervals for Expected Credit Loss (ECL)

Part 1.
Why study the topic?

Confidence Estimation of Transition Matrices is critical due to 4 key reasons

Importance

1

Regulation Compliance

2

Best practice

3

Data limitations

4

Strategic Decisions & Risk-based Pricing

Key Objectives

Importance

- Consider various approaches to confidence interval estimation and check their adequacy.
- Check if test adequacy is sensitive to imbalanced sample.
- Develop ‘rule of thumb’ for data necessary to estimate probabilities of default at certain confidence levels.
- Consider possible practical applications of confidence intervals for probabilities of default.

Part 2.

Introduction to PD Modelling using Transition Matrices

MLE-estimator for Transition Probability

General Information

Let there be 2 time periods t = {0, 1}, and state space E = {1, 2, …, n}.
Given 2 vectors of states (before and after the migration):

$$X_0 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ \dots \\ n \end{pmatrix} \qquad X_1 = \begin{pmatrix} 2 \\ 2 \\ 3 \\ \dots \\ n \end{pmatrix}$$

Then MLE-estimator for probability to migrate from state i to state j is:

$$\widehat{\mathbb{P}}(x_1 = j \mid x_0 = i) = \hat{p}_{ij}(\tau) = \frac{N_{ij}(\tau)}{N_i(\tau)},$$

where $N_{ij}(\tau)$ – number of migrations from state **i** to state **j**,
$N_i(\tau)$ – number of observations in state i initially

Transition Probabilities Comprise Markov Chain Transition Matrix

Example

General Information

Example for number of states n = 5:

X_1

X_0	1	2	3	4	5
1	90.0%	10.0%	0.0%	0.0%	0.0%
2	10.0%	80.0%	10.0%	0.0%	0.0%
3	0.0%	10.0%	80.0%	10.0%	0.0%
4	0.0%	0.0%	10.0%	80.0%	10.0%
5	0.0%	0.0%	0.0%	0.0%	100.0%

$\widehat{\mathbb{P}}(x_1 = 4 \mid x_0 = 3)$

Transition Matrix Interpretation

Example

General Information

If states are credit ratings or days past due groups, the transition matrix might comprise PD PIT / LT[1] model

Probabilities of Default (PD)

	AA	A	B	C	D
AA	90.0%	10.0%	0.0%	0.0%	0.0%
A	10.0%	80.0%	10.0%	0.0%	0.0%
B	0.0%	10.0%	80.0%	10.0%	0.0%
C	0.0%	0.0%	10.0%	80.0%	10.0%
D	0.0%	0.0%	0.0%	0.0%	100.0%

Absorbing Default state

1 PD PIT / LT – Probability of Default 'Point in Time' / Lifetime

Part 3. Experiment 1.

Adequacy Check of Confidence Interval Estimation Methods

MLE-estimator for Transition Probability

Experiment 1

- Wald Intervals[1] ?
- Bootstrap ?
- BMCMC[2] (accept – reject) ?

1 Hanson, S., & Schuermann, T. (2006). Confidence intervals for probabilities of default. Journal of Banking & Finance, 30(8), 2281-2301.

2 Bladt, M., & Sørensen, M. (2005). Statistical inference for discretely observed Markov jump processes. Journal of the Royal Statistical Society: Series B (Statistical Methodology), 67(3), 395-410.

MLE-estimator for Transition Probability

True Transition matrix

M

MLE-estimator for Transition Probability

True Transition matrix

M

Generated Sample

$$\begin{pmatrix} 1 \\ 2 \\ 2 \\ \dots \\ n \end{pmatrix} \rightarrow \begin{pmatrix} 2 \\ 2 \\ 3 \\ \dots \\ n \end{pmatrix}$$

MLE-estimator for Transition Probability

True Transition matrix

M

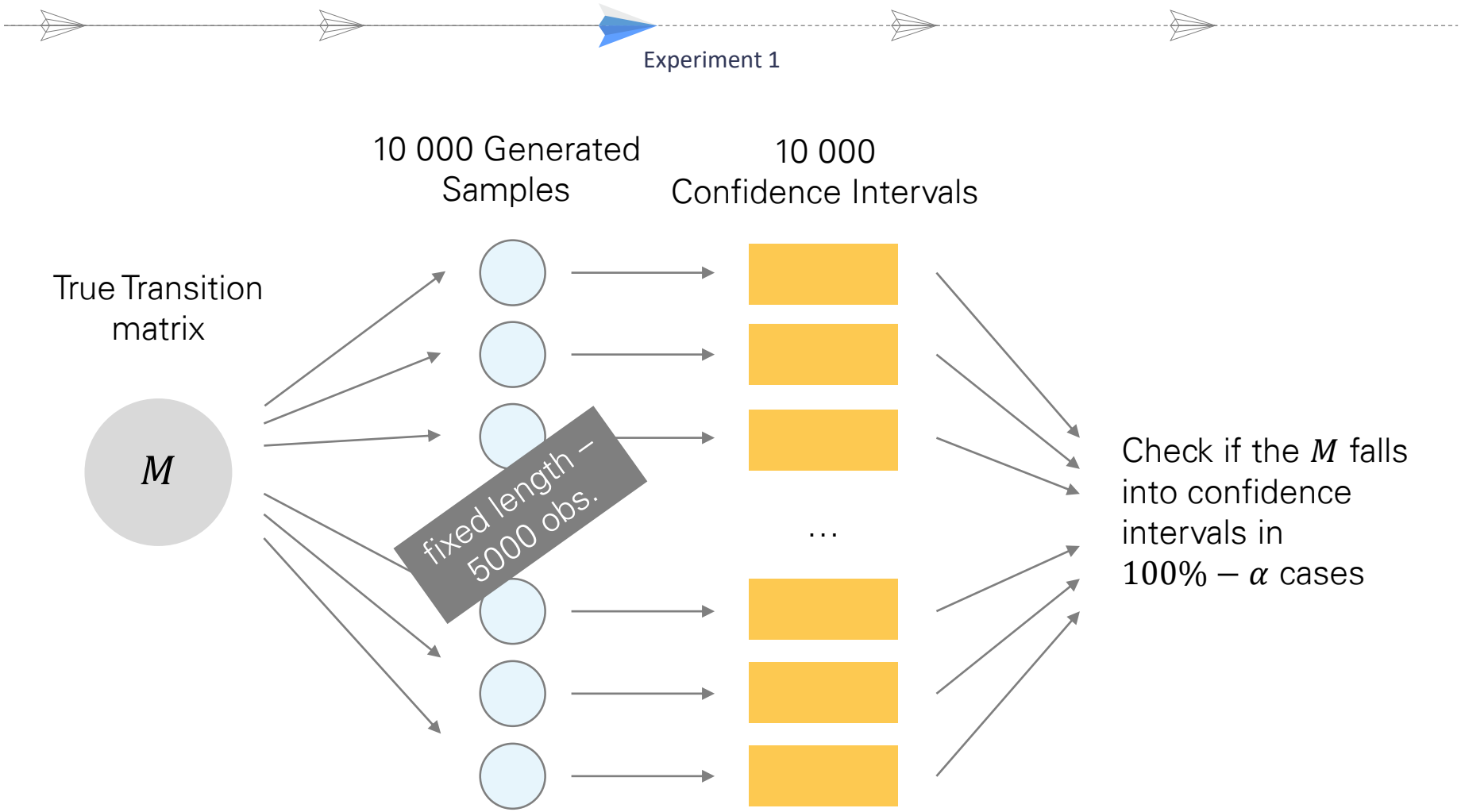
Generated Sample

$$\begin{pmatrix} 1 \\ 2 \\ 2 \\ \dots \\ n \end{pmatrix} \rightarrow \begin{pmatrix} 2 \\ 2 \\ 3 \\ \dots \\ n \end{pmatrix}$$

Confidence interval for Matrix Coefficients

$$M_{low} < M < M_{high}$$

MLE-estimator for Transition Probability



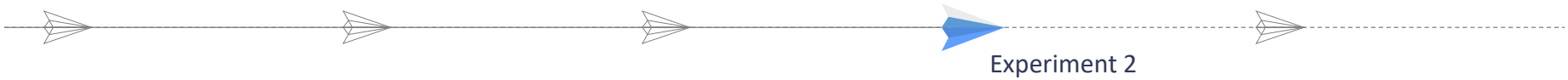
MLE-estimator for Transition Probability

Wald Intervals	Too narrow CI
Bootstrap	Adequate
BMCMC (accept – reject)	Adequate, but might be subject to model approximation error

Part 4. Experiment 2.

Data Requirements for Fixed Confidence Interval Widths

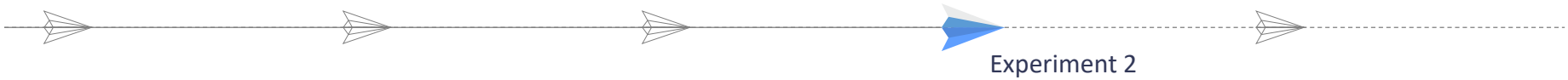
MLE-estimator for Transition Probability



True Transition matrix

M

MLE-estimator for Transition Probability



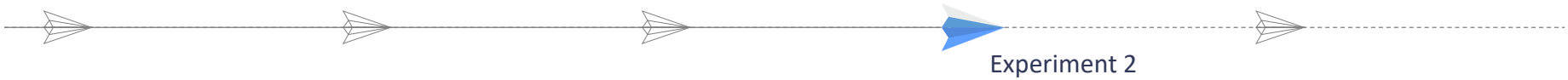
True Transition matrix

M

Generated Sample

$$\begin{pmatrix} 1 \\ 2 \\ 2 \\ \dots \\ n \end{pmatrix} \rightarrow \begin{pmatrix} 2 \\ 2 \\ 3 \\ \dots \\ n \end{pmatrix}$$

MLE-estimator for Transition Probability



True Transition matrix

M

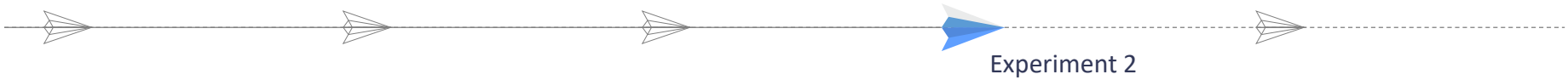
Generated Sample

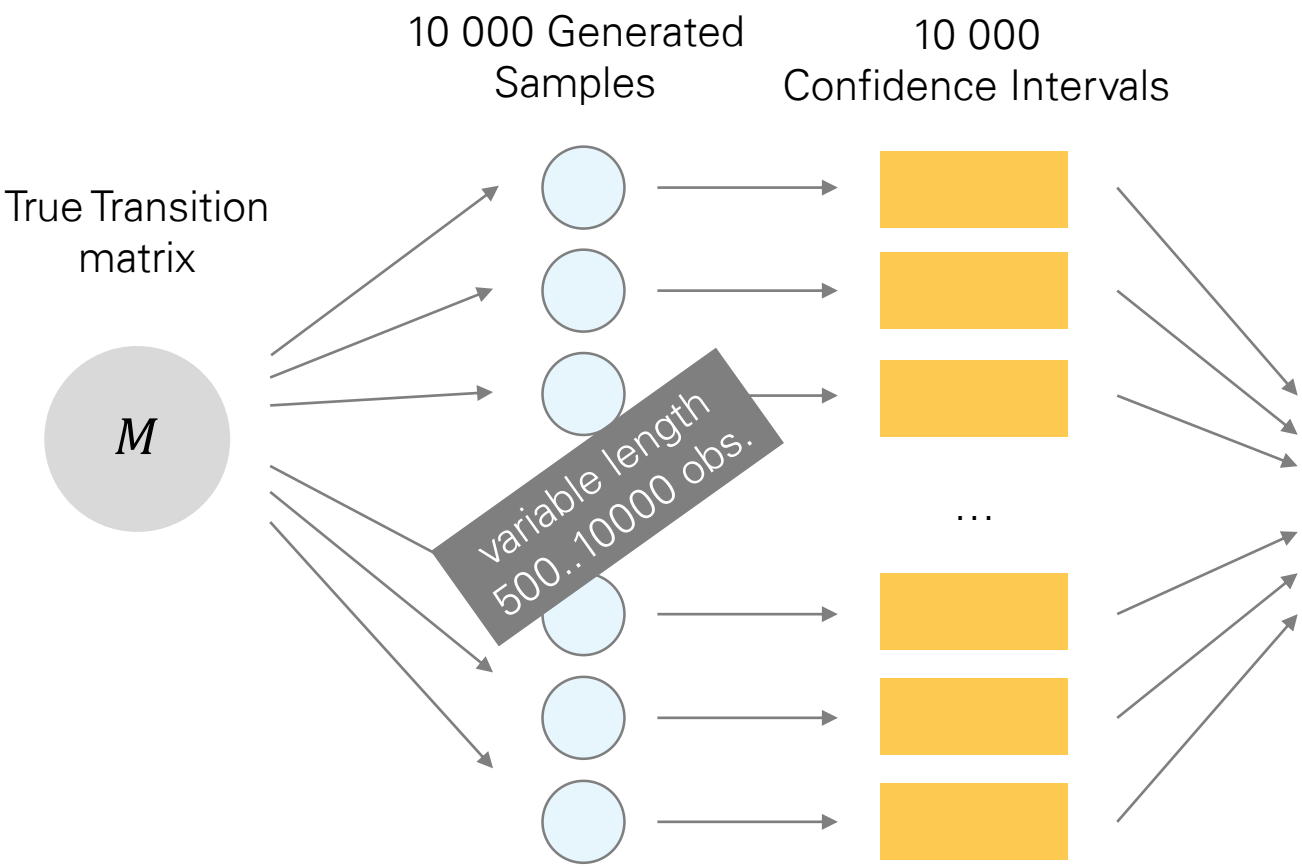
$$\begin{pmatrix} 1 \\ 2 \\ 2 \\ \dots \\ n \end{pmatrix} \rightarrow \begin{pmatrix} 2 \\ 2 \\ 3 \\ \dots \\ n \end{pmatrix}$$

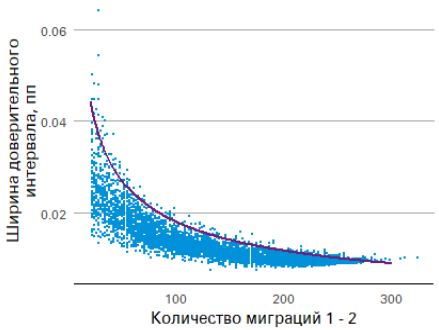
Confidence interval for Matrix Coefficients

$$M_{low} < M < M_{high}$$

MLE-estimator for Transition Probability

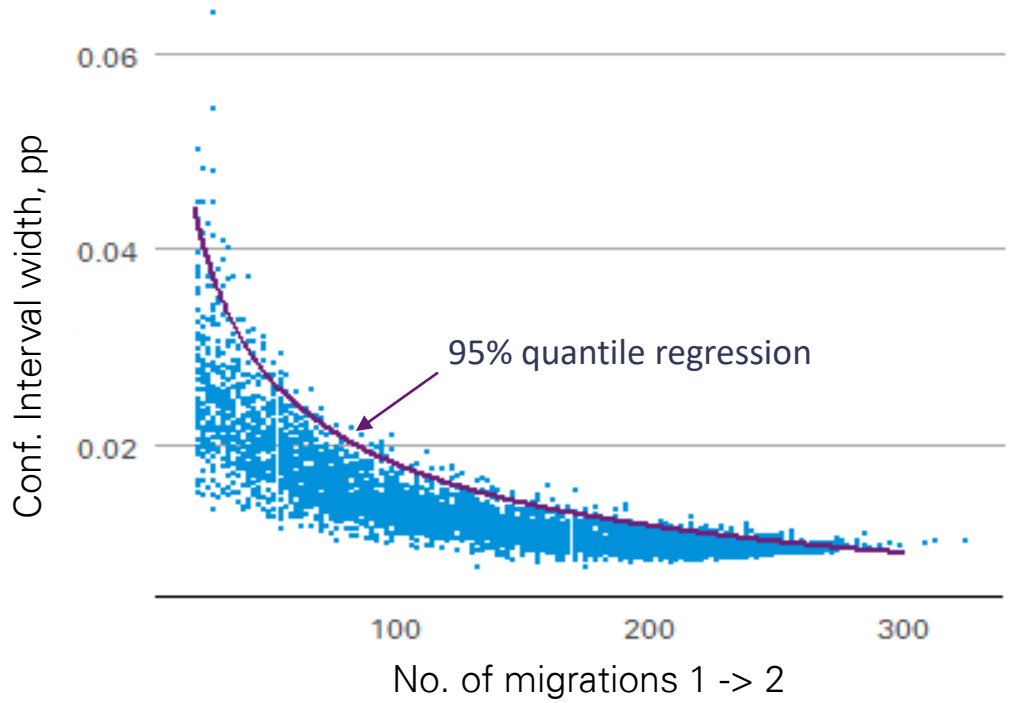






MLE-estimator for Transition Probability

Experiment 2

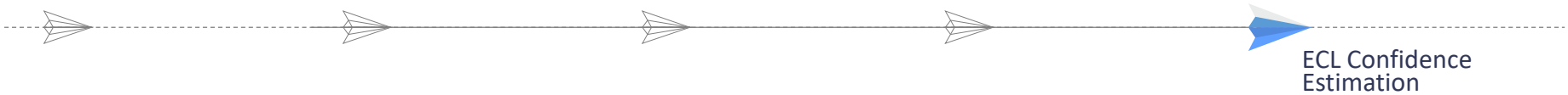


CI width, pp	(True) migration probability – Number of migration observations						
	1.0%	2.5%	5%	7.5%	90%	92.5%	95%
0.5	167	619	1 720	3 400	65 000	52 000	39 550
1.0	43	249	720	1 450	22 300	18 100	13 700
2.0	–	83	250	500	6 700	5 500	4 100
5.0	–	–	50	100	1 200	1 000	740

Part 5. Practical Example

Confidence Intervals for Expected Credit Loss (ECL)

MLE-estimator for Transition Probability



Most general formula to estimate impairment for assets:

$$ECL = PD \cdot LGD \cdot EAD \cdot D,$$

where

ECL – Expected Credit Loss,

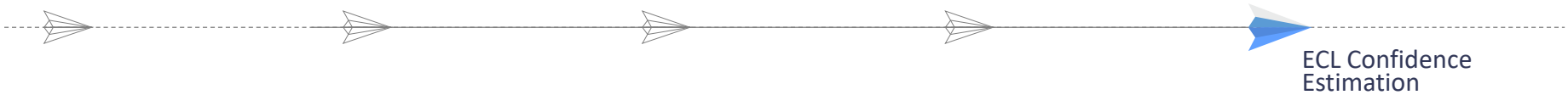
PD – Probability of Default,

LGD – Loss Given Default (assumed 65%),

EAD – Exposure at Default (assumed 85%),

D – Discount (assumed discount rate r = 10%, D = 1/(1 + r))

MLE-estimator for Transition Probability



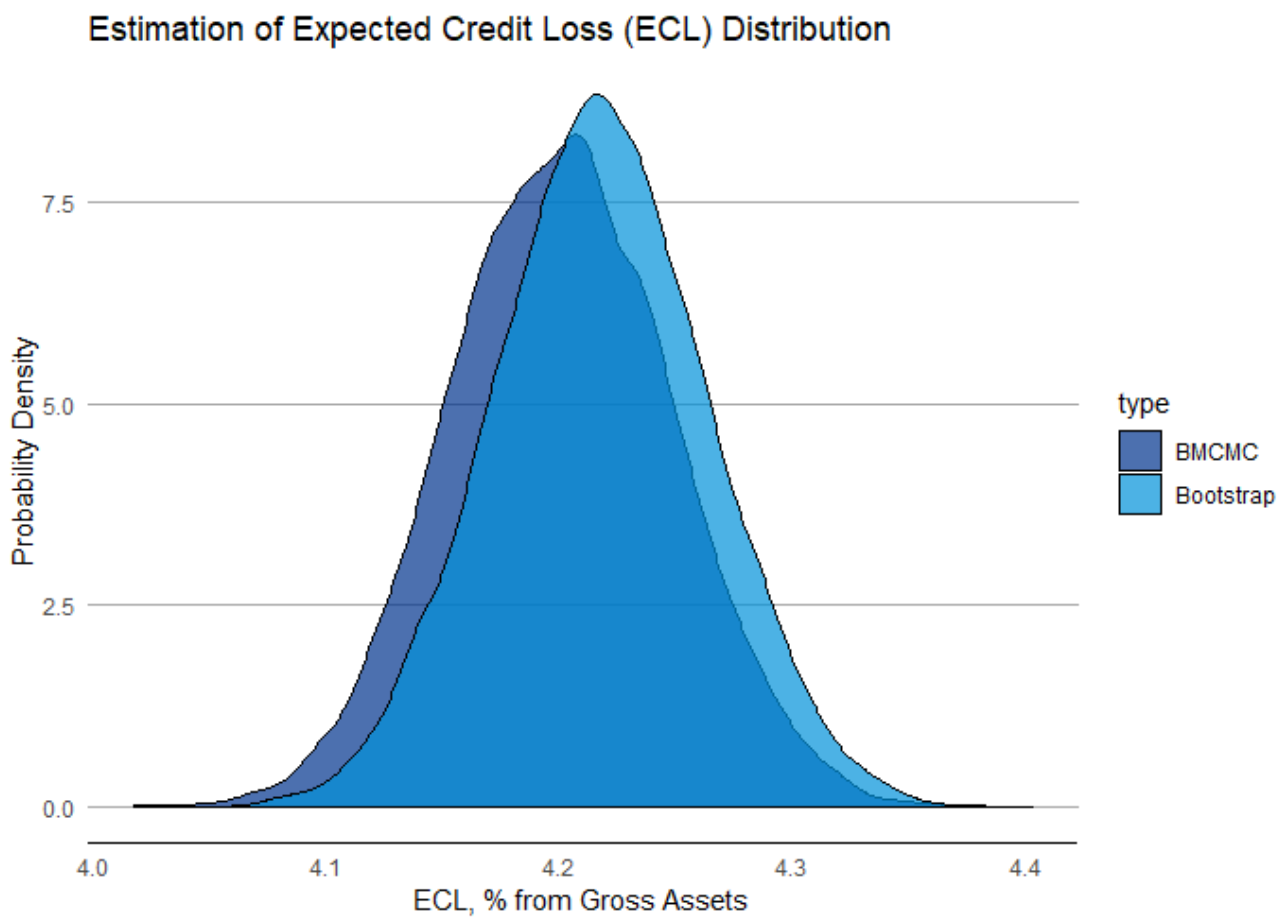
Source of data: expertly defined portfolio

Why retail?

1. Less complicated contracts in terms of payment structure.
2. No credit lines (no Credit Conversion Factor estimation).
3. Large amounts of data.
4. Traditional states for Markov chains based on Days past due (DPD):
   - DPD = 0 days
   - DPD = 1…30 days
   - DPD = 31…60 days
   - DPD = 61…90 days
   - DPD > 91 days (default)

MLE-estimator for Transition Probability

ECL Confidence Estimation



Key Results

- Analytical formula (Wald) for confidence intervals resulted to be somewhat narrow
- Bootstrap and BMCMC provide adequate confidence intervals no sensitivity to imbalanced sample found
- BMCMC might be subject to model approximation error
- ‘Rule of thumb’ for data necessary to estimate probabilities of default (PD) at certain confidence levels has been developed.
- Practical example of probabilities of default (PD) confidence intervals has been demonstrated.

Appendices

Appendix 1. Wald Confidence Intervals

Methodology [8]:

$$CI_W = \widehat{PD}_R \pm \kappa \cdot \sqrt{\frac{\widehat{PD}_R \cdot \left(1 - \widehat{PD}_R\right)}{N_R}},$$

where $\widehat{PD}_R$ – estimated migration probability,

N_R – no. of obs. in state R at the beginning of the period

κ – Normal distribution quantile $\Phi(1 - \alpha/2)$

Perks: analytical solution, speed of calculation

Cons: too narrow!

Appendix 2. Bootstrap Confidence Intervals

Sampling from the obtained data:

$$\begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \\ 4 \\ \dots \\ 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 3 \\ \dots \\ 5 \end{pmatrix}$$

1 time

2 times

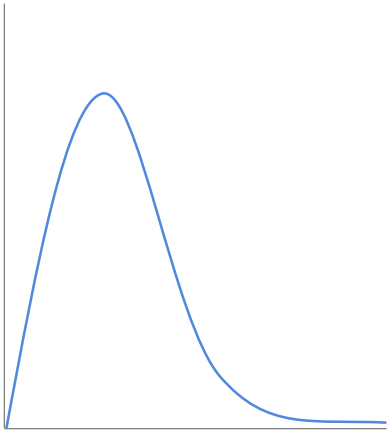
1 time

1 time

10 000 bootstrap samples:

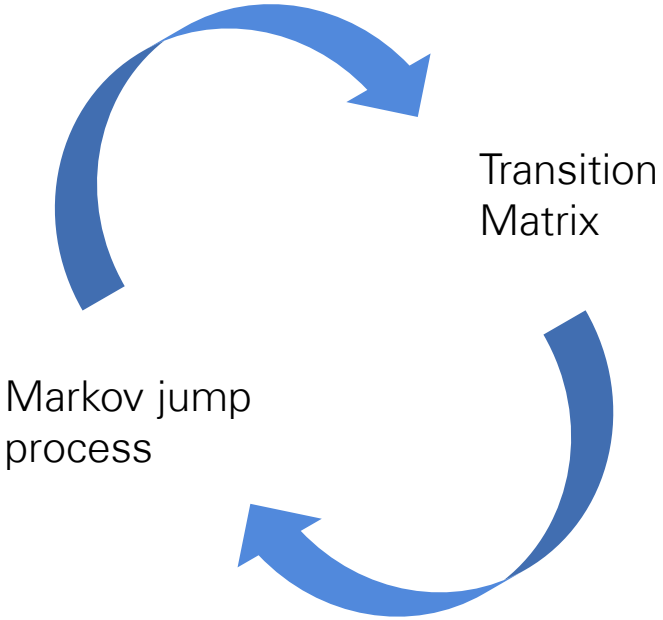
$$\begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \\ 4 \\ \dots \\ 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \\ 3 \\ \dots \\ 5 \end{pmatrix}$$

Empirical distribution:

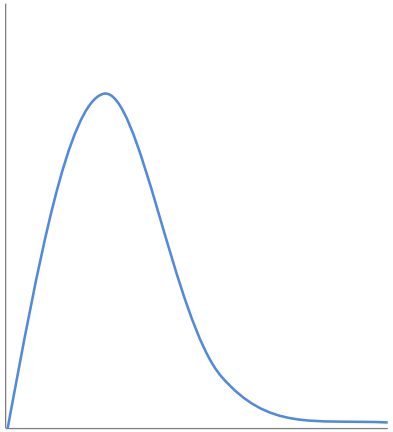


Appendix 3. BMCMC Confidence Intervals

10 000 iterations (+1000 – burn-in)



Empirical Multivariate Distribution of Matrix Coefficients



Appendix 4.
Estimation of Transition Matrices in MCMC

MLE – estimator
(Jacobsen, 1982)

$$\hat{q}_{ij}^{(c)}(\tau) = N_{ij}(\tau)/R_{ij}(\tau),$$

A posteriori estimator [1]:

$$\begin{aligned} p^*(\boldsymbol{Q}) &= L_\tau^{(c)}(\boldsymbol{Q})\ \phi(\boldsymbol{Q}) \\ &\propto \prod_{i=1}^{n} \prod_{j \neq i} q_{ij}^{N_{ij}(\tau)+\alpha_{ij}-1} e^{-q_{ij}(R_i(\tau)+\beta_i)}, \end{aligned}$$

where $N_{ij}(\tau)$ – number of transitions from state i to j in the time interval $[0, \tau]$, mark (c) stands for continuous time, and

$$R_i(t) = \int_0^t I\{X(s) = i\} ds$$

is the time spent in state i before time t.

α, β – parameters of Gamma distribution. [1] suggest using Γ(1,1).

[1] Bladt, M., & Sørensen, M. (2005). Statistical inference for discretely observed Markov jump processes. Journal of the Royal Statistical Society: Series B (Statistical Methodology), 67(3), 395-410.

Appendix 5. Markov Jump Process Simulation

1. Accept – reject algorithm [1, 2]
2. Metropolis-Hastings algorithm [1]:

$$\tilde{\boldsymbol{Q}} = \frac{1}{\alpha}\begin{pmatrix} -n & 1 & 1 & \dots & 1 & 1 \\ 1 & -n & 1 & \dots & 1 & 1 \\ . & . & . & \dots & . & . \\ 1 & 1 & 1 & \dots & -n & 1 \\ 1 & 1 & 1 & \dots & 1 & -n \end{pmatrix}$$

$$w(\boldsymbol{X}_k) = \frac{L_k^{(c)}(\boldsymbol{Q}; \boldsymbol{X}_k)\exp(\Delta\tilde{\boldsymbol{Q}})_{ij}}{L_k^{(c)}(\tilde{\boldsymbol{Q}}; \boldsymbol{X}_k)\exp(\Delta\boldsymbol{Q})_{ij}}$$

3. Bisection algorithm [3]

[1] Bladt, M., & Sørensen, M. (2009). Efficient estimation of transition rates between credit ratings from observations at discrete time points. Quantitative Finance, 9(2), 147-160.

[2] Bladt, M., & Sørensen, M. (2005). Statistical inference for discretely observed Markov jump processes. Journal of the Royal Statistical Society: Series B (Statistical Methodology), 67(3), 395-410.

[3] Asmussen, S., & Hobolth, A. (2012). Markov bridges, bisection and variance reduction. In Monte Carlo and Quasi-Monte Carlo Methods 2010 (pp. 3-22). Springer, Berlin, Heidelberg.

Appendix 6. Adequacy check - BMCMC


```
     [,1]      [,2]      [,3]      [,4]        [,5]
[1,] 0.9349169 0.9505938 0.9477435 0.001425178 0.0000000
[2,] 0.9634204 0.9515439 0.9429929 0.950593824 0.0000000
[3,] 0.9667458 0.9453682 0.9539192 0.952494062 0.9339667
[4,] 0.0000000 0.9743468 0.9496437 0.946318290 0.9524941
[5,] 0.0000000 0.0000000 0.0000000 0.000000000 0.0000000
```

Appendix 6. Adequacy check - bootstrap

	1	2	3	4	5
1	93.8%	93.7%	63.8%	100.0%	100.0%
2	94.6%	95.3%	94.2%	62.4%	100.0%
3	65.1%	94.8%	95.0%	95.0%	62.8%
4	100.0%	59.7%	95.0%	95.2%	94.3%
5	100.0%	100.0%	100.0%	100.0%	100.0%