Method of averaging in Clifford algebras and applications

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International Conference on Mathematical Methods in Physics,
April 1 - 5, Marrakesh, Morocco

Real Clifford algebra, notations

Let us consider the real Clifford algebra $C\ell_{p,q}$, $p+q=n$, with the identity element e and the generators e_a, $a=1,\dots,n$, satisfying

$$e_a e_b + e_b e_a = 2\eta_{ab} e, \qquad \eta = ||\eta_{ab}|| = \mathrm{diag}(\underbrace{1,\dots,1}_{p},\underbrace{-1,\dots,-1}_{q}). \tag{1}$$

We use notation with ordered multi-indices A for the basis elements of the Clifford algebra $C\ell_{p,q}$:

$$e_A = e_{a_1\dots a_k} = e_{a_1}\cdots e_{a_k}, \qquad 1 \le a_1 < \dots < a_k \le n. \tag{2}$$

We denote the length of multi-index A by $|A|$. In the particular case of the identity element e, we have empty multi-index $\emptyset$ of the length 0.
We denote

$$e^a := \eta^{ab} e_b = (e_a)^{-1}, \qquad e^A := (e_A)^{-1}. \tag{3}$$

We call the subspace of $C\ell_{p,q}$ of Clifford algebra elements, which are linear combinations of basis elements with multi-indices of length $|A| = k$, the subspace of grade k and denote it by $C\ell^k_{p,q}$. We denote projection operator onto the subspace of grade k by π_k, $k = 0, 1, \ldots, n$.
We denote even subspace (subalgebra) by $C\ell^{(0)}_{p,q}$ and odd subspace by $C\ell^{(1)}_{p,q}$.
We have

$$C\ell_{p,q} = \bigoplus_{k=0}^{n} C\ell^k_{p,q}, \qquad C\ell^{(j)}_{p,q} = \bigoplus_{k=j \mod 2} C\ell^k_{p,q}, \qquad j = 0, 1.$$

We denote grade involution (main involution) in Clifford algebra by

$$\widehat{U} := U|_{e_a \to -e_a}, \qquad U \in C\ell_{p,q}, \tag{4}$$

and reversion (anti-involution) by

$$\widetilde{U} := U|_{e_{a_1 \ldots a_k} \to e_{a_k} \ldots e_{a_1}}, \qquad U \in C\ell_{p,q}. \tag{5}$$

Method of averaging

Reynolds operator acting on a Clifford algebra element:

$$R_G(U) := \frac{1}{|G|} \sum_{g \in G} g^{-1} U g, \qquad U \in C\ell_{p,q}, \tag{6}$$

where $|G|$ is the number of elements in finite subgroup $G \subset C\ell_{p,q}^{\times}$.

J. D. Dixon, *Computing irreducible representations of groups*, Math. of comp. (1970).

L. Babai, K. Friedl, *Approximate representation theory of finite groups*, Found. of Comp. Science, (1991).

We can take Salingaros group $G = \{\pm e_A\}$ and obtain the following operator.

Theorem

Operator F is a projection $F^2 = F$ onto the center of Clifford algebra:

$$F(U) := \frac{1}{2^n} e_A U e^A = \pi_{\mathrm{Cen}}(U) = \begin{cases} \pi_0(U), & \text{if } n = 0 \bmod 2; \\ \pi_0(U) + \pi_n(U), & \text{if } n = 1 \bmod 2. \end{cases} \tag{7}$$

Shirokov D., *Method of Averaging in Clifford Algebras*, Advances in Applied Clifford Algebras, **27**:1 (2017) 149–163.

Shirokov D., *Contractions on Ranks and Quaternion Types in Clifford Algebras*, Vestn. Samar. Gos. Tekhn. Univ., **19**:1 (2015) 117–135.

$$\sum_{|A|=j \bmod 2} (e_A U e^A), \quad j = 0, 1, \qquad \sum_{|A|=k \bmod 4} (e_A U e^A), \quad k = 0, 1, 2, 3,$$

$$\sum_{|A|=m} (e_A U e^A), \quad m = 1, 2, \ldots, n.$$

Theorem

$$F_{\mathrm{Even}}(U) := \frac{1}{2^{n-1}} \sum_{|A|=0 \mod 2} e_A U e^A = \pi_0(U) + \pi_n(U),$$

$$F_{\mathrm{Odd}}(U) := \frac{1}{2^{n-1}} \sum_{|A|=1 \mod 2} e_A U e^A = \pi_0(U) + (-1)^{n+1}\pi_n(U).$$

These operators are projections $F^2_{\mathrm{Even}} = F_{\mathrm{Even}}$, $F^2_{\mathrm{Odd}} = F_{\mathrm{Odd}}$. If n is odd, then $F = F_{\mathrm{Even}} = F_{\mathrm{Odd}}$. If n is even, then $F = \frac{1}{2}(F_{\mathrm{Even}} + F_{\mathrm{Odd}})$.
If n is even, then

$$\pi_0(U) = \frac{1}{2^n}\Big(\sum_{|A|=0 \mod 2} e_A U e^A + \sum_{|A|=1 \mod 2} e_A U e^A\Big) = \frac{1}{2^n}\sum_A e_A U e^A,$$

$$\pi_n(U) = \frac{1}{2^n}\Big(\sum_{|A|=0 \mod 2} e_A U e^A - \sum_{|A|=1 \mod 2} e_A U e^A\Big).$$

Quaternion types

$$
\begin{aligned}
C\ell_{p,q} &= C\ell^{\overline{0}}_{p,q} \oplus C\ell^{\overline{1}}_{p,q} \oplus C\ell^{\overline{2}}_{p,q} \oplus C\ell^{\overline{3}}_{p,q}, && (8)\\
\overline{\mathbf{k}} := C\ell^{\overline{k}}_{p,q} &= \bigoplus_{m=k \mod 4} C\ell^{m}_{p,q}, \qquad k = 0,1,2,3. && (9)
\end{aligned}
$$

$$
\begin{aligned}
&[\overline{\mathbf{k}},\overline{\mathbf{k}}] \subset \overline{\mathbf{2}}, \qquad [\overline{\mathbf{k}},\overline{\mathbf{2}}] \subset \overline{\mathbf{k}}, \qquad k = 0,1,2,3; && (10)\\
&[\overline{\mathbf{0}},\overline{\mathbf{1}}] \subset \overline{\mathbf{3}}, \quad [\overline{\mathbf{0}},\overline{\mathbf{3}}] \subset \overline{\mathbf{1}}, \quad [\overline{\mathbf{1}},\overline{\mathbf{3}}] \subset \overline{\mathbf{0}};\\
&\{\overline{\mathbf{k}},\overline{\mathbf{k}}\} \subset \overline{\mathbf{0}}, \qquad \{\overline{\mathbf{k}},\overline{\mathbf{0}}\} \subset \overline{\mathbf{k}}, \qquad k = 0,1,2,3; && (11)\\
&\{\overline{\mathbf{1}},\overline{\mathbf{2}}\} \subset \overline{\mathbf{3}}, \quad \{\overline{\mathbf{1}},\overline{\mathbf{3}}\} \subset \overline{\mathbf{2}}, \quad \{\overline{\mathbf{2}},\overline{\mathbf{3}}\} \subset \overline{\mathbf{1}}.
\end{aligned}
$$

- Shirokov D.S., Classification of elements of Clifford algebras according to quaternionic types, Dokl. Math., 80:1 (2009).
- Shirokov D.S., *Quaternion typification of Clifford algebra elements*, Adv. Appl. Cliff. Alg., 22(1), 243-256, (2012).
- Shirokov D.S., *Development of the method of quaternion typification of Clifford algebra elements*, Adv. Appl. Cliff. Alg., 22(2), 483-497, (2012).

Theorem

$$\sum_{|A|=0 \mod 4} e_A U e^A = \sum_{k=0}^{3} 2^{\frac{n-2}{2}} \cos(\frac{\pi k}{2} - \frac{\pi n}{4}) \pi_{\overline{k}}(U) + 2^{n-2}(\pi_0(U) + \pi_n(U)),$$

$$\sum_{|A|=1 \mod 4} e_A U e^A = \sum_{k=0}^{3} (-1)^{k+1} 2^{\frac{n-2}{2}} \sin(\frac{\pi k}{2} - \frac{\pi n}{4}) \pi_{\overline{k}}(U) + 2^{n-2}(\pi_0(U) + (-1)^{n+1}\pi_n(U)),$$

$$\sum_{|A|=2 \mod 4} e_A U e^A = \sum_{k=0}^{3} -2^{\frac{n-2}{2}} \cos(\frac{\pi k}{2} - \frac{\pi n}{4}) \pi_{\overline{k}}(U) + 2^{n-2}(\pi_0(U) + \pi_n(U)),$$

$$\sum_{|A|=3 \mod 4} e_A U e^A = \sum_{k=0}^{3} (-1)^{k} 2^{\frac{n-2}{2}} \sin(\frac{\pi k}{2} - \frac{\pi n}{4}) \pi_{\overline{k}}(U) + 2^{n-2}(\pi_0(U) + (-1)^{n+1}\pi_n(U)).$$

If n is even:

$$\pi_{\overline{0}}(U) = 2^{\frac{-n-2}{2}} \sum_{k=0}^{3} (-1)^k \cos(\frac{\pi k}{2} - \frac{\pi n}{4}) \sum_{|A|=k \mod 4} e_A U e^A + 2^{-2}(eUe + e_{1\ldots n} U e^{1\ldots n}),$$

$$\pi_{\overline{1}}(U) = 2^{\frac{-n-2}{2}} \sum_{k=0}^{3} \sin(\frac{\pi k}{2} - \frac{\pi n}{4}) \sum_{|A|=k \mod 4} e_A U e^A + 2^{-2}(eUe - e_{1\ldots n} U e^{1\ldots n}),$$

$$\pi_{\overline{2}}(U) = 2^{\frac{-n-2}{2}} \sum_{k=0}^{3} (-1)^k \cos(\frac{\pi k}{2} - \frac{\pi n}{4}) \sum_{|A|=k \mod 4} e_A U e^A + 2^{-2}(eUe + e_{1\ldots n} U e^{1\ldots n}),$$

$$\pi_{\overline{3}}(U) = 2^{\frac{-n-2}{2}} \sum_{k=0}^{3} (-1) \sin(\frac{\pi k}{2} - \frac{\pi n}{4}) \sum_{|A|=k \mod 4} e_A U e^A + 2^{-2}(eUe - e_{1\ldots n} U e^{1\ldots n}).$$

Theorem

$$F_m(U) = \sum_{A:|A|=m} e_A U e^A = \sum_{k=0}^{n} (-1)^{km} \Big(\sum_{i=0}^{m} (-1)^i C_k^i C_{n-k}^{m-i}\Big) \pi_k(U), \quad C_n^k = \frac{n!}{k!(n-k)!}$$

$$F_1(U) = \sum_a e_a U e^a = \sum_{k=0}^{n} (-1)^k (n-2k) \pi_k(U).$$

Lounesto P., *Clifford Algebras and Spinors.* Cambridge: Cambridge Univ. Press, 306 pp. (1997).

$$\lambda_k := (-1)^k (n-2k), \qquad F_1^0(U) = U, \qquad F_1^m(U) = \underbrace{F_1(F_1(\ldots F_1(}_{m} U))\ldots), \quad (12)$$

$$\begin{pmatrix} F_1^0(U) \\ F_1^1(U) \\ \ldots \\ F_1^n(U) \end{pmatrix} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \lambda_0 & \lambda_1 & \ldots & \lambda_n \\ \ldots & \ldots & \ldots & \ldots \\ (\lambda_0)^n & (\lambda_1)^n & \ldots & (\lambda_n)^n \end{pmatrix} \begin{pmatrix} \pi_0(U) \\ \pi_1(U) \\ \ldots \\ \pi_n(U) \end{pmatrix},$$

Theorem 1) If n is even, then $A_{(n+1)\times(n+1)} = ||a_{km}||$, $a_{km} = (\lambda_{m-1})^{k-1}$, is invertible, and

$$\pi_k(U) = \sum_{m=0}^{n} b_{km} F_1^m(U), \qquad B = ||b_{km}|| = A^{-1}.$$

2) If n is odd, then A is not invertible, but $D_{\frac{n+1}{2}\times\frac{n+1}{2}} = ||d_{km}||$, $d_{km} = (\lambda_{m-1})^{k-1}$, is invertible, and

$$\pi_k(U) + \pi_{n-k}(U) = \sum_{m=0}^{\frac{n-1}{2}} g_{km} F_1^m(U), \qquad G = ||g_{km}|| = D^{-1}.$$

N.G. Marchuk, D.S. Shirokov, *General solutions of one class of field equations*, Rep. Math. Phys., 78:3 (2016), 305–326, arXiv:1406.6665

D.S. Shirokov, *Covariantly constant solutions of the Yang-Mills equations*, Adv. Appl. Cliff. Alg., 28 (2018), 53, 16 pp., arXiv:1709.07836

Example: $n = 2$

$$
\begin{aligned}
F_1^0(U) &= U = \pi_0(U) + \pi_1(U) + \pi_2(U), \\
F_1^1(U) &= 2\pi_0(U) - 2\pi_2(U), \qquad F_1^2(U) = 4\pi_0(U) + 4\pi_2(U).
\end{aligned}
$$

$$
A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 4 & 0 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{8} \\ 1 & 0 & -\frac{1}{4} \\ 0 & -\frac{1}{4} & \frac{1}{8} \end{pmatrix},
$$

$$
\begin{aligned}
\pi_0(U) &= \frac{1}{4}F_1^1(U) + \frac{1}{8}F_1^2(U) = \frac{1}{4}e^a U e_a + \frac{1}{8}e^a e^b U e_b e_a, \\
\pi_1(U) &= U - \frac{1}{4}e^a e^b U e_b e_a, \qquad \pi_2(U) = -\frac{1}{4}e^a U e_a + \frac{1}{8}e^a e^b U e_b e_a.
\end{aligned}
$$

"Commutator" equations

Theorem

Let an element $X \in C\ell_{p,q}$ satisfy the system of 2^n equations with some given elements $Q_A \in C\ell(p,q)$

$$e_A X + \epsilon X e_A = Q_A, \qquad \forall A \in \mathrm{I} = \{\emptyset, 1, \ldots, n, 12, \ldots, 1\ldots n\}, \quad \epsilon \in \mathbb{R}^\times. \tag{13}$$

If $\epsilon = -1$ (commutator case), then this system of equations either has no solution or it has a unique solution up to element of the center:

$$X = -\frac{1}{2^n} Q_A e^A + Z, \qquad Z \in \mathrm{Cen}(C\ell_{p,q}). \tag{14}$$

If $\epsilon \neq -1$, then this system of equations either has no solution or it has a unique solution

$$X = \begin{cases} \frac{1}{2^n \epsilon}\left(Q_A e^A - \frac{1}{(\epsilon+1)}\pi_0(Q_A e^A)\right), & \text{if } n \text{ is even}, \\ \frac{1}{2^n \epsilon}\left(Q_A e^A - \frac{1}{(\epsilon+1)}(\pi_0(Q_A e^A) + \pi_n(Q_A e^A))\right), & \text{if } n \text{ is odd}. \end{cases} \tag{15}$$

Example: spin connection.

Let us consider a set of smooth functions $h_a : \mathbb{R}^{k,l} \to C\ell_{p,q}$

$$h_a(x) = y_a(x)e + y_a^b(x)e_b + \cdots + y_a^{1\ldots n}(x)e_{1\ldots n} = y_a^A(x)e_A, \tag{16}$$

which satisfy conditions

$$h_a(x)h_b(x) + h_b(x)h_a(x) = 2\eta_{ab}e, \qquad a, b = 1, \ldots, n, \qquad \forall x \in \mathbb{R}^{k,l}. \tag{17}$$

Let us consider the system of equations for unknown $C_\mu : \mathbb{R}^{k,l} \to C\ell_{p,q}$

$$\partial_\mu h_a - [C_\mu, h_a] = 0, \qquad \mu = 1, \ldots, m = k + l, \qquad a = 1, \ldots, n = p + q. \tag{18}$$

From the system (18), it follows that (Lemma)

$$\partial_\mu h_A - [C_\mu, h_A] = 0, \qquad \mu = 1, \ldots, m, \qquad \forall A. \tag{19}$$

The system (18) has a unique solution $C_\mu : \mathbb{R}^{k,l} \to C\ell_{p,q} \setminus \mathrm{Cen}(C\ell_{p,q})$

$$C_\mu = \frac{1}{2^n}(\partial_\mu h_A)h^A, \qquad \mu = 1, \ldots, m. \tag{20}$$

In the case of odd n, the expression (20) can be represented as

$$C_\mu = \frac{1}{2^{n-1}} \sum_{|A|=1}^{\frac{n-1}{2}} (\partial_\mu h_A)h^A, \qquad \mu = 1, \ldots, m. \tag{21}$$

In the particular case $h_a(x) = y_a^b(x)e_b \in \mathbb{R}^{k,l} \to C\ell^1_{p,q}$, we obtain the standard formula for spin connection

$$C_\mu = \frac{1}{4}(\partial_\mu h_a)h^a : \mathbb{R}^{k,l} \to C\ell^2_{p,q}. \tag{22}$$

Conjugate action on Clifford algebras

$$(e_A)^{-1} U e_A = \sum_B m_{AB} \pi_{e_B}(U),$$

$$M_{2^n \times 2^n} = ||m_{AB}||,\ m_{AB} = m_{BA} = (e_A)^{-1} e_B e_A (e_B)^{-1} = \begin{cases} 1, & \text{if } [e_A, e_B] = 0; \\ -1, & \text{if } \{e_A, e_B\} = 0, \end{cases}$$

$$N_{2^{n-1} \times 2^{n-1}} = ||n_{AB}||,\ n_{AB} = m_{AB}.$$

Lemma

The matrices M and N are symmetric $M^T = M, \quad N^T = N$. The matrix M is invertible in the case of even n, $M^{-1} = \frac{1}{2^n} M$, and is not invertible in the case of odd n. The matrix N is invertible in the case of odd n, $N^{-1} = \frac{1}{2^{n-1}} N$.

If n is even: $\quad \pi_{e_A}(U) = \dfrac{1}{2^n} \displaystyle\sum_B m_{AB} (e_B)^{-1} U e_B.$

If n is odd: $\quad \pi_{e_A}(U) + \pi_{e_A^\star}(U) = \dfrac{1}{2^{n-1}} \displaystyle\sum_{|B| \le \frac{n-1}{2}} n_{AB} (e_B)^{-1} U e_B, \qquad e_A e_A^\star = \pm e_{1 \ldots n}$

Generalization of the method of averaging

Consider Clifford algebra $C\ell_{p,q}$ with 2 different sets γ_a, β_a, $a = 1, 2, \ldots, n$,

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}e, \qquad \beta_a\beta_b + \beta_b\beta_a = 2\eta_{ab}e. \tag{23}$$

$$\tau(F) = \frac{1}{2^n}\sum_A \beta_A F \gamma^A, \qquad \sigma(G) = \frac{1}{2^n}\sum_A \gamma_A G \beta^A. \tag{24}$$

Lemma

Operator τ is a projection $\tau^2 = \tau$. We have $\beta_B\tau(F) = \tau(F)\gamma_B$, $\forall B$.

Lemma

In the case of even n, we have

$$\sigma(G)\tau(F) = \tau(F)\sigma(G) = \pi_0(G\tau(F)) = \pi_0(F\sigma(G)).$$

In the case of odd n, we have

$$\sigma(G)\tau(F) = \tau(F)\sigma(G) = \pi_0(G\tau(F)) + \pi_n(G\tau(F)) = \pi_0(F\sigma(G)) + \pi_n(F\sigma(G)).$$

Pauli's fundamental theorem

Theorem (Pauli, 1936)

Consider 2 sets of square complex matrices γ_a, β_a, $a = 1, 2, 3, 4$ of size 4

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\delta_{ab}\mathbf{1}, \qquad \beta_a\beta_b + \beta_b\beta_a \quad = \quad 2\delta_{ab}\mathbf{1}. \tag{25}$$

Then there exists a unique (up to multiplication by a complex constant) complex matrix T such that $\gamma_a = T^{-1}\beta_a T, \qquad a = 1, 2, 3, 4.$

- W.Pauli, *Contributions mathematiques a la theorie des matrices de Dirac*, Ann. Inst. Henri Poincare 6, (1936).
- D. S. Shirokov, *Extension of Pauli's theorem to Clifford algebras*, Dokl. Math., 84:2 (2011), 699-701.
- D.S. Shirokov, *Calculations of elements of spin groups using generalized Pauli's theorem*, Adv. Appl. Cliff. Alg., 25:1 (2015), arXiv: 1409.2449
- N.G. Marchuk, D.S. Shirokov, *Local generalization of Pauli's theorem*, 2018, 16 pp., arXiv:1201.4985

Generalization of Pauli theorem

Theorem Consider real (or complexified) Clifford algebra $C\ell_{p,q}$ of dimension $n = p + q$. Let the following 2 sets of Clifford algebra elements γ_a, β_a, $a = 1, 2, \ldots, n$ satisfy conditions

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}e, \quad \beta_a\beta_b + \beta_b\beta_a = 2\eta_{ab}e. \tag{26}$$

Even case (n - even): Then both sets of elements generate bases of Clifford algebra and there exists a unique (up to multiplication by a real (complex) constant) element $T \in C\ell_{p,q}$ such that

$$\gamma_a = T^{-1}\beta_a T, \qquad \forall a = 1, \ldots, n. \tag{27}$$

Moreover, we can always find this element T in the form $T = \sum_A \beta_A F \gamma^A$, where F is any element of a set

1) $\{\gamma_A, |A| = 0 \mod 2\}$ if $\beta_{1\ldots n} \neq -\gamma_{1\ldots n}$; 2) $\{\gamma_A, |A| = 1 \mod 2\}$ if $\beta_{1\ldots n} \neq \gamma_{1\ldots n}$

such that corresponding T is nonzero $T \neq 0$.

Odd case (n - odd): Then, in the case of Clifford algebra $C\ell_{p,q}$ of signature $p-q \equiv 1 \bmod 4$, elements $\gamma_{1\ldots n}$ and $\beta_{1\ldots n}$ equals $\pm e_{1\ldots n}$ and then corresponding sets generate bases of Clifford algebra or equals $\pm e$ and then corresponding sets don't generate bases.
In the case of Clifford algebra $C\ell_{p,q}$ of signature $p-q \equiv 3 \bmod 4$, elements $\gamma_{1\ldots n}$ and $\beta_{1\ldots n}$ equals $\pm e_{1\ldots n}$, and then corresponding sets generate bases of Clifford algebra or (only for complex Clifford algebra) equals $\pm ie$ and then corresponding sets don't generate bases.
There exists a unique (up to an invertible element of Clifford algebra center) element T such that

$$
\begin{array}{lllll}
1) & \gamma_a = T^{-1}\beta_a T, & \forall a = 1,\ldots,n & \Leftrightarrow & \beta_{1\ldots n} = \gamma_{1\ldots n}, \\
2) & \gamma_a = -T^{-1}\beta_a T, & \forall a = 1,\ldots,n & \Leftrightarrow & \beta_{1\ldots n} = -\gamma_{1\ldots n}, \\
3) & \gamma_a = e_{1\ldots n}T^{-1}\beta_a T, & \forall a = 1,\ldots,n & \Leftrightarrow & \beta_{1\ldots n} = e_{1\ldots n}\gamma_{1\ldots n}, \\
4) & \gamma_a = -e_{1\ldots n}T^{-1}\beta_a T, & \forall a = 1,\ldots,n & \Leftrightarrow & \beta_{1\ldots n} = -e_{1\ldots n}\gamma_{1\ldots n}, \\
5) & \gamma_a = ie_{1\ldots n}T^{-1}\beta_a T, & \forall a = 1,\ldots,n & \Leftrightarrow & \beta_{1\ldots n} = ie_{1\ldots n}\gamma_{1\ldots n}, \\
6) & \gamma_a = -ie_{1\ldots n}T^{-1}\beta_a T, & \forall a = 1,\ldots,n & \Leftrightarrow & \beta_{1\ldots n} = -ie_{1\ldots n}\gamma_{1\ldots n}.
\end{array}
$$

Note, that all 6 cases can be written in the form $\gamma_a = (\beta_{1\ldots n}\gamma^{1\ldots n})T^{-1}\beta_a T$.

Moreover, in the case of real Clifford algebra of signature $p - q \equiv 3 \bmod 4$, we can always find this element T in the form

$$\sum_{|A|=0 \mod 2} \beta_A F \gamma^A, \tag{28}$$

where F is any element of the set

$$\{\gamma_A, \qquad A : |A| = 0 \mod 2\}, \tag{29}$$

such that corresponding T is nonzero $T \neq 0$.
In the case of real Clifford algebra of signature $p - q \equiv 1 \bmod 4$ and complexified Clifford algebra, we can always find this element T in the form (28), where F is one of the elements of the set

$$\{\gamma_A + \gamma_B, \qquad |A| = 0 \mod 2, \quad |B| = 0 \mod 2\}. \tag{30}$$

Spin groups

We denote by $M^{\times}$ the subset of invertible elements of the set M.
The following group is called Lipschitz group

$$\Gamma^{\pm}_{p,q} = \{S \in C\ell^{(0)\times}_{p,q} \cup C\ell^{(1)\times}_{p,q} : S^{-1} C\ell^{1}_{p,q} S \subset C\ell^{1}_{p,q}\} = \{v_1 \cdots v_k : v_1, \ldots, v_k \in C\ell^{1\times}_{p,q}\}.$$

Let us consider the following subgroup of Lipschitz group

$$\Gamma^{+}_{p,q} := \{S \in C\ell^{(0)\times}_{p,q} : S^{-1} C\ell^{1}_{p,q} S \subset C\ell^{1}_{p,q}\} = \{v_1 \cdots v_{2k} : v_1, \ldots, v_{2k} \in C\ell^{1\times}_{p,q}\} \subset \Gamma^{\pm}_{p,q}.$$

The following groups are called spin groups:

$$\begin{aligned}
&\mathrm{Pin}(p,q) := \{S \in \Gamma^{\pm}_{p,q} : \widetilde{S}S = \pm e\} = \{S \in \Gamma^{\pm}_{p,q} : \widehat{\widetilde{S}}S = \pm e\},\\
&\mathrm{Pin}_{+}(p,q) := \{S \in \Gamma^{\pm}_{p,q} : \widehat{\widetilde{S}}S = +e\},\\
&\mathrm{Pin}_{-}(p,q) := \{S \in \Gamma^{\pm}_{p,q} : \widetilde{S}S = +e\},\\
&\mathrm{Spin}(p,q) := \{S \in \Gamma^{+}_{p,q} : \widetilde{S}S = \pm e\} = \{S \in \Gamma^{+}_{p,q} : \widehat{\widetilde{S}}S = \pm e\},\\
&\mathrm{Spin}_{+}(p,q) := \{S \in \Gamma^{+}_{p,q} : \widetilde{S}S = +e\} = \{S \in \Gamma^{+}_{p,q} : \widehat{\widetilde{S}}S = +e\}.
\end{aligned} \tag{31}$$

Let us consider the twisted adjoint representation

$$\psi : C\ell_{p,q}^{\times} \to \operatorname{End} C\ell_{p,q}, \quad S \to \psi_S, \quad \psi_S(U) = S^{-1} U \widehat{S},\ U \in C\ell_{p,q}.$$

The following homomorphisms are surjective with the kernel $\{\pm 1\}$:

$$\psi : \mathrm{Pin}(p,q) \to \mathrm{O}(p,q), \qquad \psi : \mathrm{Spin}(p,q) \to \mathrm{SO}(p,q)$$
$$\psi : \mathrm{Spin}_+(p,q) \to \mathrm{SO}_+(p,q), \qquad \psi : \mathrm{Pin}_+(p,q) \to \mathrm{O}_+(p,q)$$
$$\psi : \mathrm{Pin}_-(p,q) \to \mathrm{O}_-(p,q).$$

It means that

$$\text{for any } P = ||p_b^a|| \in \mathrm{O}(p,q) \text{ there exists } \pm S \in \mathrm{Pin}(p,q) : S^{-1} e_a \widehat{S} = p_a^b e_b \quad (32)$$

and for the other groups similarly. We can say that these spin groups are two-sheeted coverings of the corresponding orthogonal groups.
Details about all 5 spin groups and 5 orthogonal groups:

I. M. Benn, R. W. Tucker, *An introduction to Spinors and Geometry with Applications in Physics*, Bristol, 1987.

D. Shirokov, *Clifford algebras and their applications to Lie groups and spinors*, Lectures, 19 Int. Conf. on Geometry, Integrability and Quantization, Bulgaria (2018), 11–53, arXiv:1709.06608

Hestenes method, the case $(p, q) = (1, 3)$

For each element $P = ||p^a_b|| \in \mathrm{SO}_+(1,3)$ there exist two elements $\pm S \in \mathrm{Spin}_+(1,3)$ such that

$$S^{-1} e_a S = p^b_a e_b. \tag{33}$$

The elements $\pm S$ can be found in the following way

$$S = \pm \frac{\widetilde{L}}{\sqrt{\widetilde{L}L}}, \qquad L := p^b_a e_b e^a. \tag{34}$$

Other methods (using exponentials and exterior exponentials):

- Hestenes D., *Space-Time Algebra*, Gordon and Breach, New York 1966.
- Hestenes D., Sobczyk G., *Clifford Algebra to Geometric Calculus*, Reidel Publishing Company, Dordrecht Holland, 1984.
- Lounesto P., *Clifford Algebras and Spinors*, Cambridge Univ. Press, Cambridge 2001.
- Doran C. and Lasenby A., *Geometric Algebra for Physicists*, Cambridge University Press, Cambridge 2003.
- Marchuk N., *Parametrisations of elements of spinor and orthogonal groups using exterior exponents*, Adv. Appl. Cliff. Alg., 21:3 (2011), 583–590, arXiv:0912.5349

Theorem

Let us consider the real Clifford algebra $C\ell_{p,q}$ with even $n = p + q$. Let $P \in \mathrm{SO}(p,q)$ be an orthogonal matrix such that

$$M := \sum_{A,B} p_A^B e_B e^A \neq 0 \qquad (\Leftrightarrow \pi_{\mathrm{Cen}}(S) \neq 0). \tag{35}$$

Then we can find the elements $\pm S \in \mathrm{Spin}(p,q)$ that correspond to $P = ||p_a^b|| \in \mathrm{SO}(p,q)$ as two-sheeted covering $S^{-1} e_a S = p_a^b e_b$ in the following way:

$$S = \pm \frac{\widetilde{M}}{\sqrt{\alpha\, \widetilde{M} M}}, \qquad \text{where } \widetilde{M} M \in \mathrm{Cen}(C\ell_{p,q}) = C\ell_{p,q}^0 \cong \mathbb{R}, \tag{36}$$

and the sign

$$\alpha := \mathrm{sign}(p_{1\ldots p}^{1\ldots p}) e = \mathrm{sign}(p_{p+1\ldots n}^{p+1\ldots n}) e = \widetilde{S} S = \pm e \tag{37}$$

depends on the component of the orthogonal group $\mathrm{SO}(p,q)$ (or the corresponding component of the group $\mathrm{Spin}(p,q)$).

Theorem

Let us consider the real Clifford algebra $C\ell_{p,q}$ with odd $n = p + q$. Let $P \in \mathrm{O}(p,q)$ be an orthogonal matrix such that

$$M := \sum_{A,B} (\det P)^{|A|} p^B_A e_B e^A \neq 0 \qquad (\Leftrightarrow \pi_{\mathrm{Cen}}(S) \neq 0). \tag{38}$$

Then we can find the elements $\pm S \in \mathrm{Pin}(p,q)$ that correspond to $P = ||p^b_a|| \in \mathrm{O}(p,q)$ as two-sheeted covering $S^{-1} e_a \widehat{S} = p^b_a e_b$ in the following way:

$$S = \pm \frac{\widetilde{M}}{\sqrt{\alpha\, \widetilde{M} M}}, \tag{39}$$

where

$$\widetilde{M} M \in C\ell^0_{p,q} \subset \mathrm{Cen}(C\ell_{p,q}) \cong \begin{cases} \mathbb{R} \oplus \mathbb{R}, & \text{if } p - q = 1 \mod 4; \\ \mathbb{C}, & \text{if } p - q = 3 \mod 4, \end{cases} \tag{40}$$

$$\alpha := \begin{cases} \mathrm{sign}(p^{p+1\ldots n}_{p+1\ldots n}) e = \widetilde{S} S = \pm e, & \text{if } n = 1 \mod 4; \\ \mathrm{sign}(p^{1\ldots p}_{1\ldots p}) e = \widehat{\widetilde{S}} S = \pm e, & \text{if } n = 3 \mod 4. \end{cases} \tag{41}$$

Example: $n = 3$.

$$P \in \mathrm{SO}(3): \quad \det P = 1, \qquad S \in \mathrm{Spin}(3): \quad \alpha = \widehat{\widetilde{S}}S = \widetilde{S}S = +e,$$
$$M = \sum_{A,B} (\det P)^{|A|} p_A^B e_B e^A = e + p_a^b e_b e^a + p_{a_1 a_2}^{b_1 b_2} e_{b_1 b_2} e^{a_1 a_2} + e_{123} e^{123},$$

$$M = 2(e + p_a^b e_b e^a) = 2(e + \beta_a e^a), \qquad S = \pm \frac{\widetilde{M}}{\sqrt{\widetilde{M}M}}. \tag{42}$$

Doran C. and Lasenby A., *Geometric Algebra for Physicists*, Cambridge University Press, Cambridge 2003.

Generalization to the case of arbitrary $n = p + q$ for the rotor $S \in \mathrm{Spin}_+(p, q)$:

$$\widetilde{S} e_a S = \beta_a = p_a^b e_b, \quad S = \pm \frac{\widetilde{M}}{\sqrt{\widetilde{M}M}}, \quad M = \beta_A e^A = p_A^B e_B e^A. \tag{43}$$

D.S.Shirokov, *Calculation of elements of spin groups using method of averaging in Clifford's geometric algebra*, arXiv:1901.09405.

Thank you for your attention!