# On Multidimensional Dirac–Hestenes Equation

Dmitry Shirokov

Joint work with: Sofia Rumyantseva

HSE University, Moscow, Russia

**e-mail: dm.shirokov@gmail.com**

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# Matrix formalism

## Pseudo-Euclidean space $\mathbb{R}^{1,n}$

Notations:

- $\{x^\mu\}_{\mu=0}^n$ are Cartesian coordinates;
- $\partial_\mu = \partial/\partial x^\mu$ is a partial derivative.

## Matrices $\{\gamma^\mu\}_{\mu=0}^n$

The metric tensor $\mathbb{R}^{1,n}$ is given by the diagonal matrix $\eta$:

$$\eta = (\eta^{\mu\nu})_{\mu,\nu=0}^n = \mathrm{diag}(1, -1, -1, \ldots, -1). \tag{1}$$

Let $\{\gamma^\mu\}$ be matrices that satisfy the following anticommutation relations:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{I}, \quad \mu, \nu \in \{0, 1, \ldots, n\}, \tag{2}$$

where $\mathbb{I}$ is the identity matrix of size $2^{[(n+1)/2]}$.

# Matrix formalism

## Dirac equation

Multidimensional Dirac equation for a particle in an electromagnetic field has the form:

$$\sum_{\mu=0}^{n} i\gamma^{\mu}(\partial_{\mu} + ia_{\mu}(x))\varphi(x) = m\varphi(x).$$

## Notations

- $i$ is the imaginary unit;
- $m$ is the mass of a particle;
- $\mathbf{a}(x) = (a_0(x), \ldots, a_n(x)) : \mathbb{R}^{1,n} \to \mathbb{R}^{n+1}$ is the electromagnetic vector-potential;
- $\varphi(x) : \mathbb{R}^{1,n} \to \mathbb{C}^{2^{[(n+1)/2]}}$ is a Dirac spinor.

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# Real geometric algebra $\mathcal{G}_{1,n}$

## Generators

The generators $e^0, e^1, \ldots, e^n$ satisfy the following anticommutation relations:

$$e^\mu e^\nu + e^\nu e^\mu = 2\eta^{\mu\nu} e, \quad \mu, \nu \in \{0, 1, \ldots, n\}, \tag{3}$$

where $\eta$ is diagonal matrix (1) and $e$ is the identity element.

**Note**: relations (2) and (3) are similar

## Basis

The basis of $\mathcal{G}_{1,n}$ consists of all possible ordered products of the generators:

$$e^{\mu_1} e^{\mu_2} \ldots e^{\mu_k} = e^{\mu_1 \mu_2 \ldots \mu_k}, \quad 0 \leq \mu_1 < \mu_2 < \cdots < \mu_k \leq n.$$

We denote multi-index by $M$:

$$M = \mu_1 \mu_2 \ldots \mu_k, \quad |M| = k.$$

# Basis decomposition

## Real geometric algebra $\mathcal{G}_{1,n}$

The basis decomposition of an element $U \in \mathcal{G}_{1,n}$ is:

$$U = \sum_M u_M e^M, \quad u_M \in \mathbb{R}.$$

The dimension of $\mathcal{G}_{1,n}$:

$$\dim \mathcal{G}_{1,n} = 2^{n+1}.$$

## Complexified geometric algebra $\mathcal{G}_{1,n}$

The basis decomposition of an element $U \in \mathbb{C} \otimes \mathcal{G}_{1,n}$ is:

$$U = \sum_M u_M e^M, \quad u_M \in \mathbb{C}.$$

The dimension of $\mathbb{C} \otimes \mathcal{G}_{1,n}$:

$$\dim \mathbb{C} \otimes \mathcal{G}_{1,n} = 2^{n+2}$$

# Even and odd elements

## Even subalgebra

Let $\mathcal{G}_{1,n}^{(0)}$ be an even subalgebra of $\mathcal{G}_{1,n}$:

$$\mathcal{G}_{1,n}^{(0)} = \{U \in \mathcal{G}_{1,n} | U = \sum_{|M|=2k} u_M e^M\}, \quad \dim \mathcal{G}_{1,n}^{(0)} = 2^n.$$

An element of the even subalgebra $\mathcal{G}_{1,n}^{(0)}$ is called an even element.

## Odd subspace

Let $\mathcal{G}_{1,n}^{(1)}$ be an odd subspace of $\mathcal{G}_{1,n}$:

$$\mathcal{G}_{1,n}^{(1)} = \{U \in \mathcal{G}_{1,n} | U = \sum_{|M|=2k-1} u_M e^M\}, \quad \dim \mathcal{G}_{1,n}^{(1)} = 2^n.$$

An element of the odd subspace $\mathcal{G}_{1,n}^{(1)}$ is called an odd element.

# Left ideal

## Hermitian conjugation

In $\mathbb{C} \otimes \mathcal{G}_{1,n}$, the operation has the form:

$$U^{\dagger} = e^0 U^* e^0,$$

where the star denotes the superposition of reversion and complex conjugation:

$$U^* = \sum_M (-1)^{\frac{|M|(|M|-1)}{2}} \bar{u}_M e^M, \quad u_M \in \mathbb{C}, \qquad \widetilde{U} = \sum_M (-1)^{\frac{|M|(|M|-1)}{2}} u_M e^M.$$

## Idempotent

An element $t \in \mathbb{C} \otimes \mathcal{G}_{1,n}$ is called a Hermitian idempotent if:

$$t^2 = t, \quad t^{\dagger} = t.$$

The corresponding left ideal $L(t)$ generated by $t$:

$$L(t) = \{U \in \mathbb{C} \otimes \mathcal{G}_{1,n} | Ut = U\}.$$

# Primitive Hermitian idempotent

## Definition

If a left ideal $L(t)$ does not contain another left ideal except itself and $L(0)$, then it is called a **minimal left ideal**.

The corresponding Hermitian idempotent $t$ is called the **primitive Hermitian idempotent**.

## Idempotent in $\mathbb{C} \otimes \mathcal{G}_{1,n}$

Case $n = 3$:

$$t = \frac{1}{4}(e + e^0)(e + ie^{12}) \in \mathbb{C} \otimes \mathcal{G}_{1,3}. \tag{4}$$

Multi-dimensional case:

$$t = \frac{1}{2}(e + e^0) \prod_{\mu=1}^{[\frac{n+1}{2}]-1} \frac{1}{2}(e + ie^{2\mu-1}e^{2\mu}) \in \mathbb{C} \otimes \mathcal{G}_{1,n}. \tag{5}$$

# Spinors in different cases

## Matrix representation

$$\mathbb{C} \otimes \mathcal{G}_{1,n} \simeq \begin{cases} \mathrm{Mat}(2^d, \mathbb{C}), & n = 2d - 1 \\ \mathrm{Mat}(2^d, \mathbb{C}) \oplus \mathrm{Mat}(2^d, \mathbb{C}), & n = 2d \end{cases}$$

## Dirac spinor

A Dirac spinor is an element of the left ideal $L(t)$:

$$\varphi(x) : \mathbb{R}^{1,n} \to L(t).$$

Cases of $\varphi(x)$:

1. Spinor: $n = 2d - 1$; (5) — a **primitive** Hermitian idempotent
2. Semi-spinor: $n = 2d$; $t$ — a **primitive** Hermitian idempotent
3. Double spinor: $n = 2d$; (5) — a Hermitian idempotent

# Matrix representation of spinors

## Case $n = 2d - 1$; Spinor

$$t = \left(\begin{array}{c|c} \begin{matrix} 1 \\ 0 \\ 0 \\ \dots \end{matrix} & 0 \end{array}\right)$$

## Case $n = 2d$

| Semi-spinor | Double spinor |
|---|---|
| $t = \left(\begin{array}{ccc\|c} 1 & 0 & \dots & \\ 0 & 0 & \dots & 0 \\ & \dots & & \\ \hline & 0 & & 0 \end{array}\right)$ | $t = \left(\begin{array}{ccc\|ccc} 1 & 0 & \dots & & & \\ 0 & 0 & \dots & & 0 & \\ & \dots & & & & \\ \hline & & & 1 & 0 & \dots \\ & 0 & & 0 & 0 & \dots \\ & & & & \dots & \end{array}\right)$ |

## Dirac equation

Dirac equation for a particle in an electromagnetic field has the form:

$$\sum_{\mu=0}^{n} ie^{\mu}(\partial_{\mu} + ia_{\mu}(x))\varphi(x) = m\varphi(x), \tag{6}$$

where:

- $e^{\mu}$ is the generator of $\mathcal{G}_{1,n}$;
- $m$ is the mass of a particle;
- $\mathbf{a}(x) = (a_0(x), \ldots, a_n(x)) : \mathbb{R}^{1,n} \to \mathbb{R}^{n+1}$ is the electromagnetic vector-potential.

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# Dirac–Hestenes equation

## Lemma on unique decomposition

$$\forall \varphi \in L(t) \quad \exists! \Psi \in \mathcal{G}_{1,3}^{(0)} : \quad \varphi = \Psi t.$$

The real dimensions of the minimal left ideal $L(t)$ that is generated by (4) and the even real subalgebra $\mathcal{G}_{1,3}^{(0)}$ are the same:

$$\dim L(t) = \dim \mathcal{G}_{1,3}^{(0)} = 8.$$

## Dirac–Hestenes equation

$$\sum_{\mu=0}^{3} e^{\mu}(\partial_{\mu}\Psi(x) + \Psi(x)a_{\mu}(x)I)E + m\Psi(x)I = 0, \quad \Psi(x) \in \mathcal{G}_{1,3}^{(0)}. \tag{7}$$

$$I = -e^{12}, \quad E = e^{0}$$

## Question

Will the Dirac–Hestenes equation be similar in multidimensional case?

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# Answer: No

## 1st reason: dimensions

If $t$ is fixed as (5), then:

$$\dim L(t) = 2^{[(n+2)/2]+1}, \quad \dim \mathcal{G}_{1,n}^{(0)} = 2^n.$$

$$[\frac{n+2}{2}] + 1 < n, \quad \forall n > 4$$

## 2nd reason: lemma on unique decomposition

- Case $n = 3$:
  If $Y \in \mathcal{G}_{1,3}^{(0)}$ and $Yt = 0$, then $Y = 0$.
- Case $n = 5$:
  If $Y = e^{12} - e^{34} \in \mathcal{G}_{1,5}^{(0)}$ and $t$ is fixed as (5), then $Yt = -it + it = 0$.

# Theorem in case $n = 2d - 1$

## Conditions

Let fix the primitive Hermitian idempotent $t$ as in formula (5) and the real algebra $Q'$ be:

$$Q' = \mathcal{G}(e^0, e^1, e^2, e^3, e^5, e^7, \ldots, e^{2d-1}) \subset \mathcal{G}_{1,2d-1}. \tag{8}$$

## Dirac–Hestenes equation

$$\sum_{\mu=0,1,2,3,5,7,\ldots,2d-1} e^\mu(\partial_\mu \Psi(x) + \Psi(x) a_\mu(x) I)E +$$
$$+ \sum_{\mu=3,5,\ldots,2d-3} (\partial_{\mu+1}\Psi(x) I - \Psi(x) a_{\mu+1}(x)) e^\mu E + m\Psi(x) I = 0, \tag{9}$$

where $I = -e^{12}$ and $E = e^0$.

$\varphi(x) \in L(t)$ is a solution of multidimensional Dirac equation (6)

$\Updownarrow$

$\Psi(x) \in Q'^{(0)}$ : $\varphi(x) = \Psi(x)t$ is a solution of multidimensional Dirac–Hestenes equation (9)

# Theorem in case $n = 2d$. Semi-spinor

## Conditions

Let fix the primitive Hermitian idempotent $t$ as:

$$t = \frac{1}{2}(e + e^0) \prod_{\mu=1}^{[\frac{2d+1}{2}]} \frac{1}{2}(e + ie^{2\mu-1}e^{2\mu}) \in \mathbb{C} \otimes \mathcal{G}_{1,2d}. \tag{10}$$

and the real algebra $Q'$ be:

$$Q' = \mathcal{G}(e^0, e^1, e^2, e^3, e^5, e^7, \ldots, e^{2d-1}) \subset \mathcal{G}_{1,2d}. \tag{11}$$

## Dirac–Hestenes equation

$$\sum_{\mu=0,1,2,3,5,7,\ldots,2d-1} e^\mu(\partial_\mu \Psi(x) + \Psi(x)a_\mu(x)I)E +$$
$$+ \sum_{\mu=3,5,\ldots,2d-3,2d-1} (\partial_{\mu+1}\Psi(x)I - \Psi(x)a_{\mu+1}(x))e^\mu E + m\Psi(x)I = 0. \tag{12}$$

# Theorem in case $n = 2d$. Semi-spinor

$\varphi(x) \in L(t)$ is a solution (semi-spinor) of multidimensional Dirac equation (6)

$$\Updownarrow$$

$\Psi(x) \in Q'^{(0)}$ : $\varphi(x) = \Psi(x)t$ is a solution of the multidimensional Dirac–Hestenes equation (12)

# Theorem in case $n = 2d$. Double spinor

## Conditions

Let fix the Hermitian idempotent $t$ as in formula (5) and the real algebra $Q'$ be:

$$Q' = \mathcal{G}(e^0, e^1, e^2, e^3, e^5, e^7, \ldots, e^{2d-3}, e^{2d-1}, e^{2d}) \subset \mathcal{G}_{1,2d}. \tag{13}$$

## Dirac–Hestenes equation

$$\sum_{\mu=0,1,2,3,5,7,\ldots,2d-1,2d} e^\mu(\partial_\mu \Psi(x) + \Psi(x) a_\mu(x) I)E +$$
$$+ \sum_{\mu=3,5,\ldots,2d-3} (\partial_{\mu+1}\Psi(x) I - \Psi(x) a_{\mu+1}(x)) e^\mu E + m\Psi(x) I = 0. \tag{14}$$

$\varphi(x) \in L(t)$ is a solution (double spinor) of multidimensional Dirac equation (6)

$$\Updownarrow$$

$\Psi(x) \in Q'^{(0)}$ : $\varphi(x) = \Psi(x)t$ is a solution of the multidimensional Dirac–Hestenes equation (14)

# Dirac–Hestenes equation for graphene

## Matrix formalism

$$\varphi(x) = \begin{pmatrix} \varphi_{A,1}(x) + i\varphi_{A,2}(x) \\ \varphi_{B,1}(x) + i\varphi_{B,2}(x) \end{pmatrix} \in \mathbb{C}^2 \Rightarrow \varphi(x) \text{ is a semi-spinor}$$

## Geometric algebra formalism

We consider Dirac equation (6) in $\mathbb{C} \otimes \mathcal{G}_{1,2}$. Let us fix the idempotent $t$ as (10):

$$t = \frac{1}{4}(e + e^0)(e + ie^1e^2) \in \mathbb{C} \otimes \mathcal{G}_{1,2}.$$

The real algebra $Q'$ is:

$$Q' = \mathcal{G}(e^0, e^1, e^2) = \mathcal{G}_{1,2}.$$

| The Dirac–Hestenes equation | The explicit form of a solution |
|---|---|
| $\sum_{\mu=0}^{3} e^\mu(\partial_\mu \Psi + \Psi a_\mu I)E + m\Psi I = 0.$ | $\Psi = 2(\varphi_{A,1}e + \varphi_{A,2}I) + 2(\varphi_{B,1}e^{01} + \varphi_{B,2}e^{01}I) \in \mathcal{G}^{(0)}_{1,2}.$ |

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# Gauge invariance

## Theorem

*Let $Q'$ be the real algebra* (8) *(or* (11) *and* (13) *respectively). If $\Psi(x) \in Q'^{(0)}$ is a solution to multidimensional Dirac–Hestenes equation* (9) *(or* (12) *and* (14) *respectively) with an electromagnetic vector-potential $\boldsymbol{a}(x)$, then $\tilde{\Psi}(x)$ is also a solution to the same multidimensional Dirac–Hestenes equation with an electromagnetic vector-potential $\tilde{\boldsymbol{a}}(x)$:*

$$\tilde{\Psi}(x) = \Psi(x)\exp(I\lambda(x)), \qquad \tilde{a}_\mu(x) = a_\mu(x) - \partial_\mu\lambda(x), \tag{15}$$

*where $\lambda(x) : \mathbb{R}^{1,n} \to \mathbb{R}$.*

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# Lorentz invariance

Let us consider Lorentz transformations of the coordinates $\{x^\mu\}$:

$$x^\mu \mapsto \hat{x}^\mu = p^\mu_\nu x^\nu, \quad P = (p^\mu_\nu)^n_{\nu,\mu=0},$$

where the matrix $P$ belongs to one of the groups

$$\mathrm{O}(1,n) = \{A \in \mathrm{Mat}(n+1,\mathbb{R}) | A^\top \eta A = \eta\}, \tag{16}$$
$$\mathrm{SO}(1,n) = \{A \in \mathrm{O}(1,n) | \det A = 1\}. \tag{17}$$

The wave function is multiplied by an element of the corresponding spin group:

$$\mathrm{Pin}(1,n) = \{T \in \mathcal{G}^{(0)\times}_{1,n} \cup \mathcal{G}^{(1)\times}_{1,n} | \forall x \in \mathcal{G}^1_{1,n}\ TxT^{-1} \in \mathcal{G}^1_{1,n};\ T^{-1} = \pm\widetilde{T}\}, \tag{18}$$
$$\mathrm{Spin}(1,n) = \{T \in \mathcal{G}^{(0)\times}_{1,n} | \forall x \in \mathcal{G}^1_{1,n}\ TxT^{-1} \in \mathcal{G}^1_{1,n};\ T^{-1} = \pm\widetilde{T}\}. \tag{19}$$

The formula

$$S^{-1} e^\mu S = p^\mu_\nu e^\nu, \qquad \nu,\mu = 0,1,\ldots,n. \tag{20}$$

establishes a one-to-two correspondence between a single matrix $P = (p^\mu_\nu)$ in the pseudo-orthogonal group and the elements $\pm S$ of the corresponding spin group.

# Lorentz invariance in Minkowski space

## Theorem

*Let* $\mu, \nu = 0, 1, 2, 3$ *and the transformations be defined as:*

$$
\begin{aligned}
x^\mu \mapsto \widehat{x^\mu} &= p^\mu_\nu x^\nu, \quad P = (p^\mu_\nu) \in \mathrm{O}(1,3), \\
\partial_\mu \mapsto \widehat{\partial}_\mu &= q^\nu_\mu \partial_\nu, \quad Q = (q^\nu_\mu) = P^{-1}, \\
a_\mu \mapsto \widehat{a}_\mu &= q^\nu_\mu a_\nu, \\
e^\mu \mapsto \widehat{e^\mu} &= e^\mu, \\
\Psi \mapsto \widehat{\Psi} &= \begin{cases} S\Psi, & S \in \mathrm{Spin}(1,3) \quad (i.e.\, P \in \mathrm{SO}(1,3)), \\ S\Psi E, & S \in \mathrm{Pin}(1,3) \setminus \mathrm{Spin}(1,3), \end{cases}
\end{aligned}
$$

*where* $P$ *and* $S$ *are related by*

$$
S^{-1} e^\mu S = p^\mu_\nu e^\nu, \qquad \nu, \mu = 0, 1, \ldots, n. \tag{21}
$$

*If* $\Psi$ *is a solution to Dirac–Hestenes equation* (7)*, then* $\widehat{\Psi}$ *is also a solution to the transformed Dirac–Hestenes equation.*

# Lorentz invariance in multidimensional case

Any element $S \in \mathcal{G}_{1,n}^{(0)}$ can be decomposed by separating elements belonging to $Q'$ as follows:

$$S = S^0 + \sum_k S_{\mu_1\,\mu_2\,\ldots\,\mu_k} e^{\mu_1\,\mu_2\,\ldots\,\mu_k}, \tag{22}$$

where $\mu_i = 4, 6, 8, \ldots$ and $S^0 \in Q'^{(0)}$. Additionally, if $k = 2s$, then $S_{\mu_1\,\mu_2\,\ldots\,\mu_k} \in Q'^{(0)}$, otherwise $S_{\mu_1\,\mu_2\,\ldots\,\mu_k} \in Q'^{(1)}$.

## Lemma

*Let the Hermitian idempotent $t$ have form (5) ((10) or (5)) and $\psi \in L(t)$. Suppose $Q'$ is defined by (8) ((11) or (13) respectively), then there uniquely exists $\Psi \in Q'^{(0)}$ such that $\psi = \Psi t$.*
*If $S \in \mathcal{G}_{1,n}^{(0)}$ and $\widehat{\psi} = S\psi$, then $\widehat{\Psi}$ corresponded to $\widehat{\psi}$ has the form:*
$\widehat{\Psi} = S^0\Psi + \sum_k S_{\mu_1\,\mu_2\,\ldots\,\mu_k}\Psi e^{\mu_1-1\,\mu_2-1\,\ldots\,\mu_k-1} I^k \in Q'^{(0)}$.
*If $S \in \mathcal{G}_{1,n}^{(1)}$ and $\widehat{\psi} = S\psi$, then $\widehat{\Psi}$ corresponded to $\widehat{\psi}$ has the form:*
$\widehat{\Psi} = \left(S^0\Psi + \sum_k S_{\mu_1\,\mu_2\,\ldots\,\mu_k}\Psi e^{\mu_1-1\,\mu_2-1\,\ldots\,\mu_k-1} I^k\right) E \in Q'^{(0)}$.

## Theorem

*Let us fix the primitive Hermitian idempotent $t$ as in formula (5) and the real algebra $Q'$ be defined by (8). Consider the following transformations:*

$$x^\mu \mapsto \widehat{x^\mu} = p^\mu_\nu x^\nu, \quad P = (p^\mu_\nu) \in \mathrm{O}(1, 2d-1),$$
$$\partial_\mu \mapsto \widehat{\partial}_\mu = q^\nu_\mu \partial_\nu, \quad Q = (q^\nu_\mu) = P^{-1},$$
$$a_\mu \mapsto \widehat{a}_\mu = q^\nu_\mu a_\nu, \qquad e^\mu \mapsto \widehat{e^\mu} = e^\mu, \qquad \mu, \nu = 0, 1, 2, \ldots, 2d-1.$$

*Also, assume that the element $S$ is related to the matrix $P$ as in (21). If $\Psi \in Q'^{(0)}$ is a solution of equation (9) and $S \in \mathrm{Spin}(1, 2d-1)$, then the function $\widehat{\Psi}$:*

$$\widehat{\Psi} = S^0 \Psi + \sum_k S_{\mu_1 \mu_2 \ldots \mu_k} \Psi e^{\mu_1 - 1\, \mu_2 - 1 \ldots \mu_k - 1} I^k \tag{23}$$

*is a solution of the transformed Dirac–Hestenes equation. If $\Psi \in Q'^{(0)}$ is a solution of equation (9) and $S \in \mathrm{Pin}(1, 2d-1) \setminus \mathrm{Spin}(1, 2d-1)$, then the function $\widehat{\Psi}$:*

$$\widehat{\Psi} = S^0 \Psi E + \sum_k S_{\mu_1 \mu_2 \ldots \mu_k} \Psi e^{\mu_1 - 1\, \mu_2 - 1 \ldots \mu_k - 1} I^k E \tag{24}$$

*is a solution of the transformed Dirac–Hestenes equation.*

## Theorem

*Let us fix the primitive Hermitian idempotent $t$ as in formula* (10) *(or* (5)*) and the real algebra $Q'$ be defined by* (11) *(or* (13) *respectively). Consider the following transformations:*

$$x^\mu \mapsto \widehat{x^\mu} = p^\mu_\nu x^\nu, \quad P = (p^\mu_\nu) \in \mathrm{SO}(1, 2d),$$
$$\partial_\mu \mapsto \widehat{\partial}_\mu = q^\nu_\mu \partial_\nu, \quad Q = (q^\nu_\mu) = P^{-1},$$
$$a_\mu \mapsto \widehat{a}_\mu = q^\nu_\mu a_\nu, \qquad e^\mu \mapsto \widehat{e^\mu} = e^\mu, \qquad \mu, \nu = 0, 1, 2, \ldots, 2d.$$

*Also, assume that the element $S$ is related to the matrix $P$ as in* (21). *If $\Psi \in Q'^{(0)}$ is a solution of equation* (12) *(or* (14) *respectively) and $S \in \mathrm{Spin}(1, 2d)$, then the function $\widehat{\Psi}$:*

$$\widehat{\Psi} = S^0 \Psi + \sum_k S_{\mu_1 \mu_2 \ldots \mu_k} \Psi e^{\mu_1 - 1 \, \mu_2 - 1 \ldots \mu_k - 1} I^k$$

*is a solution of the transformed Dirac–Hestenes equation.*

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# Conclusions

## Advantage of considering Dirac–Hestenes equation

The Dirac–Hestenes equation gives a deeper understanding of geometry in various tasks, since the considering wave function is entirely real.

## Need to consider auxiliary algebra $Q'$

$$\dim L(t) \neq \dim \mathcal{G}_{1,n}^{(0)}.$$

## Dependence on parity of $n$

Cases:

1. $n$ is odd (spinor);
2. $n$ is even (semi-spinor);
3. $n$ is even (double spinor).

## The new equations are gauge-invariant and Lorentz-invariant.

# Thank you for your attention!

## On Multidimensional Dirac–Hestenes Equation

Dmitry Shirokov

Joint work with: Sofia Rumyantseva

HSE University, Moscow, Russia

**e-mail: dm.shirokov@gmail.com**

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