Conjectures on coherent rings of low homological dimension

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Coherent rings

Let $A = \bigoplus_{m \geq 0} A_m$ be a graded associative algebra over a field k (with 1). It is *connected* if $A_0 = k$ and *standard* if, in addition, A is generated by A_1.

A (graded) ring R is called right (graded) *coherent* if each finitely generated (homogeneous) right ideal in R is finitely presented. Equivalently, the category of (graded) finitely presented modules $\operatorname{gr} R$ is abelian.

The definition is due to Chase (1960), after Serre who introduced coherent sheaves in 1955.

Examples: Noetherian rings, $k[x_1, x_2, \dots]$, $k\langle x_1, x_2, \dots \rangle$, hereditary algebras, free products of coherent rings.

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Abelian categories associated to a graded algebra

Consider a graded algebra A as a "coordinate ring of a noncommutative projective variety". Here we need:

- $\mathrm{Gr}(A)$ and $\mathrm{gr}(A)$, the categories of $\mathbb{Z}$-graded right modules and the one of finitely presented right modules
- $\mathrm{Tors}(A)$ and $\mathrm{tors}(A)$ their full subcategories of all torsion modules (= sums of finite-dimensional modules).
- The quotient category $\mathrm{Qgr}(A) := \mathrm{Gr}(A)/\mathrm{Tors}(A)$ plays the role of the of the category of *quasicoherent sheaves* on the noncommutative variety defined by A [Artin, Zhang, 1994].
- If A is *graded coherent*, one can define the abelian category of *coherent sheaves* as $\mathrm{qgr}(A) := \mathrm{gr}(A)/\mathrm{tors}(A)$ [Polishchuk, 2005]. Then we consider the bounded derived category $\mathbf{D}^b(\mathrm{qgr}(A))$.

One-relator algebras

A group defined by a single relation is coherent. Its integral group ring is coherent as well. [A. Jaikin–Zapirain, M. Linton, Annals of Math., 2025]

Conjecture

Each connected graded algebra $A = F/(r)$ (where $F = k\langle x_1, \dots, x_n \rangle$) defined by a single homogeneous relation r is graded coherent.

Evidence to support.

- *Theorem [P., 2008].* Suppose that $\deg r = 2$. Then $A = F/(r)$ is graded coherent.
- *Proposition* Suppose that r is a monomial. Then $A = F/(r)$ is coherent.
- *Corollary.* For each homogeneous $r \in F$, there is $r' \in F$ such that the algebras $A = F/(r)$ and $A' = F/(r')$ have equal Hilbert and Poincare series, and A' is coherent.

Example [P., 2001] For $r = xy^2 - x^2y$, the algebra $A = F/(x)$ is coherent.

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Regular algebras

A graded algebra A is called regular if it has finite global dimension (say, d) and satisfies the following Gorenstein property:

$$\mathrm{Ext}^{\,i}_A(k_A, k_A) \cong \begin{cases} 0, & i \neq d \\ k[l] \text{ for some } l \in \mathbb{Z}, & i = d. \end{cases}$$

It is a noncommutative version of $\mathbb{P}^{d-1}$.
Regular algebras of polynomial growth are called Artin–Schelter (AS) regular. For $d \leq 3$, all AS regular algebras are classified (beginning with [Artin, Schelter, 1987]).

Regular algebras and coherence

Conjecture ([Artin, Schelter, 1987]) All AS regular algebras are Noetherian (hence, coherent).

Conjecture (A. Bondal) Let A be a regular algebra. Then it is graded coherent.

Theorem [a number of papers, beginning with AS, 1987] In all known cases (including all AS regular algebras with $d = 3$), AS regular algebras are Noetherian .

Theorem [P., 2008] Regular algebras of global dimension ≤ 2 are coherent.

Counter-example [Gélinas, 2018] There exists a non-coherent regular algebra with $d = 4$.

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Regular algebras and coherence: conjecture and examples

Conjecture (Bondal conjecture in dimension 3)

Regular algebras of global dimension 3 *are coherent.*

Examples: the octonion algebra [P. Smith, 2011] the Yang–Mills algebra [Herscovich, 2012], and 3-Calabi–Yau algebras which are Ore extensions of 2-Calabi–Yau ones [He, Oystaeyen, Zhang, 2013].

Proposition Suppose A is a standard regular of global dimension $d = 3$. Then there exists a graded coherent regular algebra A' of the same Hilbert and Poincare algebra as A.

Proof. Bocklandt (2008) proposed a class of examples of graded standard Calabi–Yau algebras with $d = 3$ for all possible Hilbert series. All these algebras occur to be graded coherent.

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Preprojective algebras

If Q is a finite quiver (i.e., a directed multigraph with loops), and $\overline{Q}$ is its double (which extends Q by adding, for each arrow a in Q, an oppositely directed arrow a^*), then the preprojective algebra Π_Q is defined as the quotient of the path algebra $k\overline{Q}$ by the single relation $e = \sum_{a \in Q_1} (aa^* - a^*a)$. The grading is defined by assigning degree 1 to all arrows and degree 0 to the vertices. This algebra is finite-dimensional for Dynkin quivers and Noetherian for extended Dynkin quivers. All such algebras have global dimension 2.

Conjecture

For any finite quiver, the preprojective algebra Π_Q is graded coherent.

Preprojective algebra: evidence

For example, for the quiver S with a single vertex and n loops $x_1, \dots, x_n$, the preprojective algebra

$$\Pi = k\langle x_1, \dots, x_n, x_1^*, \dots, x_n^* \mid \sum_{i=1}^{n} [x_i, x_i^*] \rangle$$

is regular and has global dimension 2, so it is coherent. It is the common case of Conjectures 1,2,3.

Theorem [Minamoto, 2012] Suppose Q is acyclic. Introduce the grading on Π_Q as follows: for each arrow a of Q, let $\deg a = 0, \deg a^* = 1$. In this grading, the algebra Π_Q is graded coherent. (And the corresponding category of coherent sheaves is derived equivalent to $\mod -kQ$).

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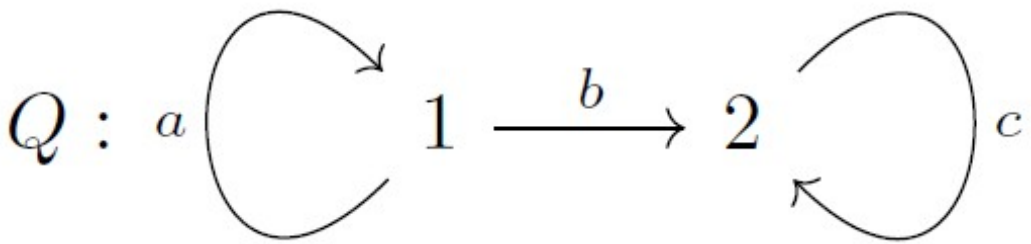
Preprojective algebras: positive result

Proposition. Suppose that a quiver Q is obtained from the disjoint union of extended Dynkin quivers by adding edges. Then its preprojective algebra Π_Q is graded coherent.

For example, the simplest pair of extended Dynkin quivers (that is, single loops) yields, after adding an edge, the quiver



Its preprojective algebra $\Pi_Q = k\overline{Q}/([a, a^*] + bb^*, [c, c^*] + b^*b)$ is graded coherent.

The monomial algebra is coherent

A f.g. graded algebra is called *right Groebner finite* if it admits a finite Groebner basis of relations and each finitely generated right-sided ideal in it has finite Groebner basis.
Facts [Lu, P., 2013; P., 1996]:

- Each finitely presented monomial algebra (= a quotient of a path algebra by some paths) is Groebner finite.
- Each right Groebner finite algebra is right coherent.

We conclude:

Theorem

Each finitely presented monomial algebra is right coherent.

Main trick to prove coherence

Theorem [P., 2008] Suppose A is a graded algebra. Let $B = A/I$, where the graded algebra B is right Noetherian and the homogeneous ideal $I \neq A$ is projective as a left A-module. Then the algebra A is right graded coherent.

Corollary Let A is graded and $B = A/I$ is *noncommutative relative complete intersection* of A (that is, I is generated by a "strongly free" set [Anick, 1987]). If B is right Noetherian, then A is right coherent.

Publications

- D. P., *Coherent algebras and noncommutative projective lines*, Journal of Algebra, 2008, **319**, 8, 3280–3290
- Lu Li, D. P. *Entropy of monomial algebras and derived categories*. International Mathematics Research Notices, 2023 (3), 2446–2473. Better version: arXiv:2103.03946

Thanks for following the coherence!