NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS

Faculty of Economic Sciences
Department of Applied Economics

The GARCH-M model with an asymmetric risk premium: Distinguishing between ‘good’ and ‘bad’ volatility periods

† **Juri Trifonov,** Research Associate, M.A. Student, Bachelor of Economics, NRU HSE (Moscow)
† **Bogdan Potanin,** Senior Lecturer, PhD in Economics, NRU HSE (Moscow)

2nd International Conference on Econometrics and Business Analytics – **iCEBA 2022**

Yerevan & Dilijan, Republic of Armenia, 2022

CEBA lab
SAINT-PETERSBURG STATE UNIVERSITY

Introduction (I): Why is GARCH-M so important?

Motivation

Volatility modelling is a key for the implementation of risk management policies

GARCH processes *(Bollerslev, 1986)*

Conditional volatility dynamics

Portfolio theories *(Markowitz, 1952), (Sharp, 1964)*

Asset returns depend on its level of risk

Volatility and return have a positive relation

How to estimate the risk premium?

The development of the GARCH-M *(Engle et al., 1987)* model

The GARCH-M process

Let's denote y as a $\mathrm{T}\times 1$ vector of log returns and σ as a $\mathrm{T}\times 1$ vector of conditional volatilities, where $\mathrm{T} \in \mathrm{N}$. Then the GARCH-M (1,1) process is specified as follows:

$$
\begin{gathered}
y_t = \mu + \overbrace{\lambda\sigma_t}^{\text{Risk premium}} + \varepsilon_t,\\
\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2,\\
\xi_t \sim \mathcal{N}(0,1)\ \ i.i.d.,\\
\varepsilon_t = \sigma_t\xi_t,
\end{gathered}
$$

where parameters ω, α, and β determine the dynamics of conditional variance and $\mathrm{t} \in \{1, \dots, \mathrm{T}\}$.

Volatility is allowed to influence return directly

Based on portfolio theories, λ is expected to be positive

Introduction (II): Leverage effect

The classic GARCH model is symmetric, i.e., it reacts equally to positive and negative shocks in returns.

Classic approach

Volatility reacts differently to positive and negative shocks in returns

Volatility reacts more sharply on negative shocks in returns rather than positive ones

The development of asymmetric GARCH models

- EGARCH (Nelson, 1991), GJR-GARCH (Glosten et al., 1993)

GJR-GARCH process

$$y_t = \mu + \varepsilon_t,$$

$$\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \gamma I_{t-1} \varepsilon_{t-t}^2 + \beta \sigma_{t-1}^2,$$

$$I_t = \begin{cases} 0, \text{ if } \varepsilon_t \geq 0 \\ 1, \text{ if } \varepsilon_t < 0 \end{cases},$$

$$\xi_t \sim \mathcal{N}(0,1) \ \ i.i.d.,$$

$$\varepsilon_t = \sigma_t \xi_t.$$

The main problem

Leverage effect may arise not only in the variance equation of the process.

The GJR-GARCH may be easily transformed into the GARCH-M-GJR process, capturing the asymmetry in the variance equation.

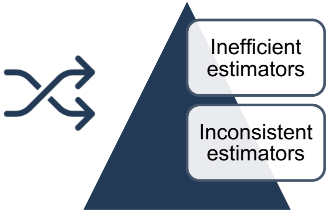
But what if risk premium itself reflects asymmetric responses to volatility changes?

The GARCH-M assumption: investors demand **similar risk premiums** during 'good' and 'bad' volatility periods.

According to (Bollerslev, 2022), there has been found statistical evidence in favor of:

- Insignificant risk premiums in asset returns
- Negative estimates of a risk premium

Inconsistent with portfolio theories!



Main contribution & novelty

In this study we propose the GARCH-M-GJR-LEV model that allows capturing the asymmetry effect both in variance and return equations.

(Bollerslev, 2022) indicates that inconsistent results in previous studies may have been caused by the presence of the leverage effect in the risk premium.

Leverage effect

Variance

Risk premium

Volatility tends to react more sharply on negative shocks in returns

Investors tend to demand a higher risk premium during 'bad' volatility periods rather than positive ones

What are 'good' and 'bad' volatility periods?

?

Volatility

'Good'

Volatility

'Bad'

Volatility

Although volatility rises in both cases, ***risk premium will not be treated the same*** *by investors!*

It is important to distinguish between 'good' and 'bad' volatility periods while modelling the risk premium!

The GARCH-M-GJR-LEV model

1. Captures the leverage effect in the **risk premium**
2. Captures the leverage effect in the **variance equation**

Consistent and efficient estimators of the risk premium

The GARCH-M-GJR-LEV model

We construct the model in the framework of the GJR-GARCH process.

Deviations:
- Conditional **variance** instead of volatility
- **Previous** period instead of current

GARCH-M-GJR-LEV

Asymmetric Risk premium

$$y_t = \mu + \lambda_1 \sigma_{t-1}^2 + \overbrace{\lambda_2 I_{t-1} \sigma_{t-1}^2} + \varepsilon_t,$$

$$\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \gamma I_{t-1} \varepsilon_{t-t}^2 + \beta \sigma_{t-1}^2,$$

$$I_t = \begin{cases} 0, \text{ if } \varepsilon_t \geq 0 \\ 1, \text{ if } \varepsilon_t < 0 \end{cases},$$

$$\varepsilon_t = \sigma_t \xi_t,$$

$$\xi_t \sim \mathcal{N}(0,1) \ i.i.d.,$$

where parameters λ_1 and λ_2 define the influence of conditional variance on returns.

Estimation via the MLE

Volatility caused by negative shocks can lead to a higher risk premium

*To impose the necessary stationarity conditions of the process, we need to specify the **unconditional variance of returns.***

The Unconditional Variance Theorem

Theorem 1. If the process is stationary, then the expression for the unconditional variance of returns in the GARCH-M-GJR-LEV model has the following form:

$$Var(y_t) = \sigma^2 = \left(\lambda_1^2 + \lambda_1 \lambda_2\right) \times \left(\mathbb{E}\left[\sigma_t^4\right] - \mathbb{E}\left[\sigma_t^2\right]^2\right) + \\ + \frac{1}{2}\lambda_2^2 \times \left(\mathbb{E}\left[\sigma_t^4\right] - \frac{1}{2}\mathbb{E}\left[\sigma_t^2\right]^2\right) + \mathbb{E}\left[\sigma_t^2\right],$$

where $\mathbb{E}\left[\sigma_t^4\right]$ is an expectation of σ_t^4 and is defined as follows:

$$\mathbb{E}\left[\sigma_t^4\right] = \frac{\omega^2 + \omega \mathbb{E}\left[\sigma_t^2\right] \times (2\alpha + 2\beta + \gamma)}{1 - 3\alpha^2 - \beta^2 - \frac{3}{2}\gamma^2 - 2\alpha\beta - 3\alpha\gamma - \beta\gamma}.$$

Also, $\mathbb{E}\left[\sigma_t^2\right]$ denotes the unconditional variance of ε_t :

$$\sigma_\varepsilon^2 = Var(\varepsilon_t) = \frac{\omega}{1 - \alpha - \gamma/2 - \beta}.$$

The process is stationary when: $\sigma^2 = Var(y_t) > 0$

Simulated Data Analysis (I): Design

To study the properties of the obtained estimators and compare them with alternatives, we conducted a simulated data analysis.

The analysis proceeds as follows:

1. Pseudo-random sample, following the **GARCH-M-GJR-LEV data generating process**
2. Obtain **MLE estimators for 3 models**: GARCH-M, GARCH-M-GJR, GARCH-M-GJR-LEV
3. **Compare estimator properties and accuracy** using information criteria and accuracy metrics

We consider two different sets of parameter values:

Set I — *Values **frequently observed** in application of GARCH to stock returns, **leverage effect** provides **significant changes** to the data generating process*

Set II — *Values that **replicate our results** of the GARCH-M-GJR-LEV model **application to the S&P 500** index stock returns*

Table 1: Parameters of simulations.

Parameter	Set I	Set II
μ	0.01	0.05
ω	0.1	0.05
α	0.1	0.05
β	0.7	0.8
λ_1	0.2	-0.05
γ	0.15	0.2
λ_2	0.5	0.2
Sample size	1000	
Number of simulations	100	

Sample size: 1000 observations

- Financial time series are **long enough**
- Larger series are usually subject to **structural breaks**

Number of simulations: 100

- Evidence in favor of **significant advantage**
- Higher number requires extensive **computational resources**

Optimization details:

- Likelihood function is **not necessary concave**
- **Hybrid genetic algorithm** + **BFGS** local optimizer

Evaluation metrics:

1. *We calculated **RMSE values based on the coefficient estimates** to compare the estimation accuracy among all of the models:*

$$RMSE(\hat{\theta}) = \sqrt{\frac{1}{100}\sum_{m=1}^{100}\left(\theta_m - \hat{\theta}_m\right)^2},$$

where $\theta \in \{\mu, \omega, \alpha, \beta, \lambda_1, \gamma, \lambda_2\}$ denotes a set of estimated parameters, and m is an index of each simulation.

2. *Besides that, we calculated **RMSE values** both **for predicted conditional volatilities and returns**:*

$$RMSE(\hat{\sigma}) = \frac{1}{100}\sum_{m=1}^{100}\sqrt{\frac{1}{T}\sum_{t=1}^{T}(\sigma_t - \hat{\sigma}_t)^2},$$

$$RMSE(\hat{y}) = \frac{1}{100}\sum_{m=1}^{100}\sqrt{\frac{1}{T}\sum_{t=1}^{T}(y_t - \hat{y}_t)^2}.$$

Simulated Data Analysis (II): Results

According to the simulated data analysis results, the proposed method demonstrates a significant advantage over the existing counterparts.

Coefficient accuracy results

RMSE criterion demonstrates that the proposed method provides significantly more accurate coefficient estimates than other models.

Table 2: Accuracy metrics of coefficient estimates (Set I).

Metric/Model	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
$RMSE(\hat{\mu})$	7.091	6.056	4.616
$RMSE(\hat{\omega})$	3.390	3.111	2.609
$RMSE(\hat{\alpha})$	10.106	4.013	4.144
$RMSE(\hat{\beta})$	6.147	5.885	5.403
$RMSE(\hat{\lambda}_1)$	21.438	7.657	6.639
$RMSE(\hat{\gamma})$	-	22.917	6.513
$RMSE(\hat{\lambda}_2)$	-	-	8.934

Table 3: Accuracy metrics of coefficient estimates (Set II).

Metric/Model	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
$RMSE(\hat{\mu})$	4.196	3.283	3.383
$RMSE(\hat{\omega})$	3.150	1.694	1.700
$RMSE(\hat{\alpha})$	12.824	3.092	3.168
$RMSE(\hat{\beta})$	7.106	4.095	3.965
$RMSE(\hat{\lambda}_1)$	14.072	26.764	25.596
$RMSE(\hat{\gamma})$	-	15.310	23.970
$RMSE(\hat{\lambda}_2)$	-	-	6.997

The estimators may become inaccurate without accounting for the leverage effect

The method allows estimating the λ_1 parameter more accurately

The model tends to estimate λ_2 accurately (even more precisely than λ_1)

Volatility & return accuracy results

RMSE criterion demonstrates that the proposed method provides significantly more accurate predictions of conditional volatilities and returns.

Table 4: Accuracy metrics and information criteria (Set I).

Metric/Model	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
$RMSE(\hat{\sigma})$	12.345	10.295	6.142
$Victories_\sigma\%$	0%	4%	96%
$RMSE(\hat{y})$	94.581	94.210	90.568
$Victories_y\%$	0%	0%	100%
AIC	2573.072	2559.644	2500.384
BIC	2597.610	2589.091	2534.738

Table 5: Accuracy metrics and information criteria (Set II).

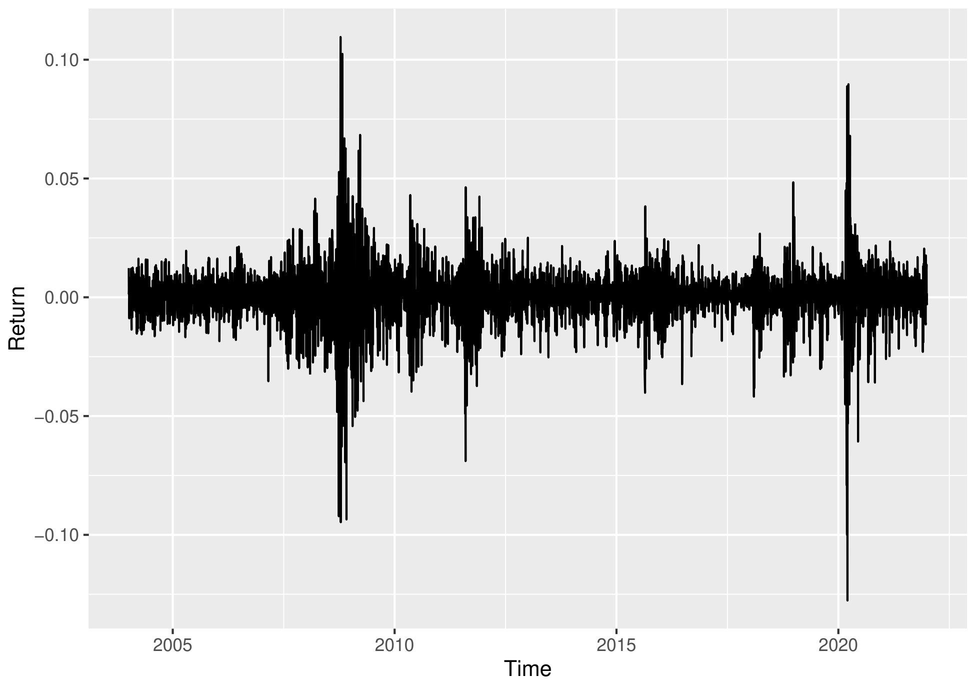
Metric/Model	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
$RMSE(\hat{\sigma})$	13.290	6.938	5.804
$Victories_\sigma\%$	0%	19%	81%
$RMSE(\hat{y})$	100.393	100.285	98.766
$Victories_y\%$	0%	4%	96%
AIC	2619.222	2598.067	2585.704
BIC	2643.761	2627.514	2620.058

*If the **data generating process** involves the **asymmetric relationship between risk premium and volatility**, then the **existing methods may demonstrate significantly lower accuracy of forecasts** for volatilities and returns.*

S&P 500 Market Index Application: Overview

The main goal of the analysis is to examine the S&P 500 index for the asymmetric responses of risk premium to conditional volatility and compare the results of the GARCH-M-GJR-LEV model with alternatives.

S&P 500 Index Returns



Data description:

Sample size: 4531 observations

- ❖ The whole sample includes 4531 observations

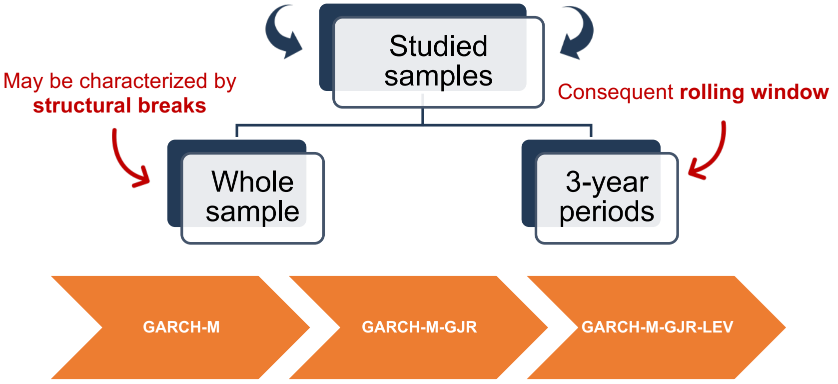
Period: 01.01.2004 – 31.12.2021

- ❖ The whole sample covers a period of 17 years

Covariance matrix

- ❖ The covariance matrix was estimated via the Gradient Outer Product (GOP)

*To ensure the robustness of the results we estimated models both for the **entire sample and specific periods**.*



S&P 500 Market Index Application: Whole sample analysis (I)

There has been found statistical evidence in favor of the presence of the asymmetric risk premium in the S&P index returns. The proposed method was able to demonstrate a significant advantage over its counterparts.

Table 6: S&P 500 estimation results for the period 2004-2021.

Parameters	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
μ	0.0499***	0.0225	0.0347**
	(0.0141)	(0.0138)	(0.0140)
ω	0.0286***	0.0287***	0.0291***
	(0.0024)	(0.0020)	(0.0012)
α	0.1347***	0.0129***	0.0162***
	(0.0084)	(0.0042)	(0.0030)
β	0.8389***	0.8589***	0.8553***
	(0.0092)	(0.0080)	(0.0005)
λ_1	0.0265*	0.0135	-0.0237***
	(0.0137)	(0.0128)	(0.0091)
γ	-	0.1903***	0.1871***
		(0.0120)	(0.0091)
λ_2	-	-	0.0760***
			(0.0003)
AIC	**11791.596**	**11643.245**	**11629.996**

Note: *** — $p < 0.01$, ** — $p < 0.05$, *– $p < 0.1$; st.errors in parentheses.

Results:

Negative sign of λ_1 — Contradicts portfolio theories!

- May be caused by structural breaks

Positive sign of λ_2

- Statistical evidence in favor of leverage effect in risk premium
- Investors demand a higher risk premium during 'bad' volatility periods

Insignificant estimate of λ_1 in GARCH-M-GJR

- Leverage coefficient in the volatility equation takes on all the influence
- By accounting for the leverage effect, one may misidentify the absence of the risk premium in returns

The lowest value of Akaike criterion

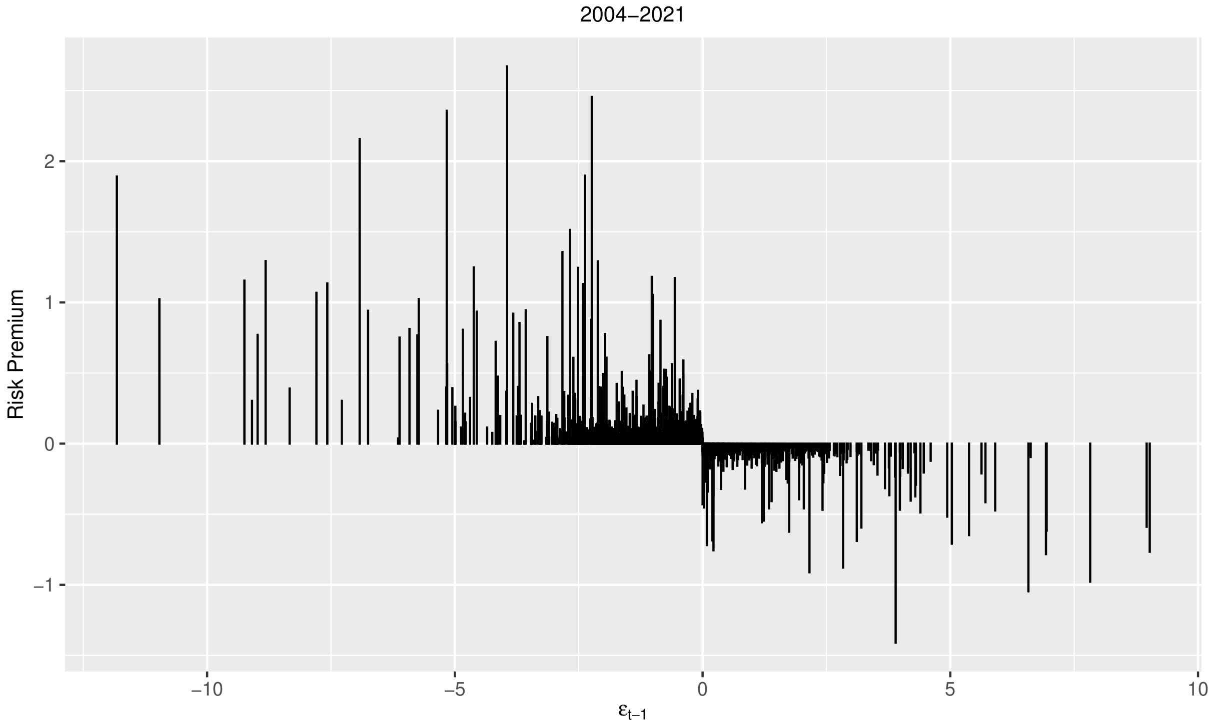
- The GARCH-M-GJR-LEV appears to be the best according to the AIC criterion

To properly identify the risk premium and leverage effects, the proposed GARCH-M-GJR-LEV model should be used because of the presence of an asymmetric risk premium effect in the observed data.

S&P 500 Market Index Application: Whole sample analysis (II)

Here we provide a graphical visualization of the dependence between shocks in returns and risk premium.

The relationship between the risk premium and shocks is returns



Interpretation

$\lambda_1 < 0$

$\lambda_2 > 0$

Risk premium only rises when shocks in returns are negative

During 'good' volatility periods, investors tend to demand a discount instead of a premium.

The graph clearly demonstrates contradictory findings

- May be caused by structural breaks

In the further slides we will discuss the results on 3-year periods

S&P 500 Market Index Application: Rolling window (I)

In this subsection we apply the model to analyze S&P 500 index returns on periods 2017-2021 using a rolling window.

Table 7: S&P 500 estimation results for the period 2019-2021.

Parameters	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
μ	0.0994***	0.0762**	0.0801**
	(0.032)	(0.0319)	(0.0327)
ω	0.0656***	0.0718***	0.0711***
	(0.0108)	(0.0115)	(0.0115)
α	0.3034***	0.1623***	0.1616***
	(0.0321)	(0.0231)	(0.023)
β	0.6777***	0.6569***	0.6574***
	(0.0296)	(0.0297)	(0.0298)
λ_1	0.0325	0.0149	-0.0114
	(0.0219)	(0.019)	(0.0271)
γ	-	0.3176***	0.3205***
		(0.0682)	(0.0682)
λ_2	-	-	0.0644
			(0.0405)
AIC	**2058.326**	**2046.602**	**2046.535**

Note: *** — $p < 0.01$, ** — $p < 0.05$, *– $p < 0.1$; st.errors in parentheses.

Table 8: S&P 500 estimation results for the period 2018-2020.

Parameters	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
μ	0.1017***	0.0780**	0.0829**
	(0.0314)	(0.0328)	(0.0328)
ω	0.0550***	0.0533***	0.0522***
	(0.0092)	(0.0094)	(0.0093)
α	0.2467***	0.1390***	0.1399***
	(0.0289)	(0.0208)	(0.0210)
β	0.7227***	0.7228***	0.7242***
	(0.0274)	(0.0261)	(0.0260)
λ_1	0.0185	0.0060	-0.0156
	(0.0242)	(0.0224)	(0.0097)
γ	-	0.2116***	0.2077***
		(0.0437)	(0.0426)
λ_2	-	-	0.0489
			(0.0317)
AIC	**2145.537**	**2132.454**	**2133.173**

Note: *** — $p < 0.01$, ** — $p < 0.05$, *– $p < 0.1$; st.errors in parentheses.

All the three periods demonstrate the **same pattern**

No statistical evidence of risk premium effects

- Statistically insignificant estimates of λ_1 and λ_2
- The finding coincides with GARCH-M and GARCH-M-GJR
- Insignificant difference in AIC criterion

Statistical evidence of leverage effect in the variance equation

- Statistically significant estimate of γ
- Volatility reacts more sharply on negative shocks
- Significant difference in AIC criterion between GARCH-M and other models

GJR-GARCH

S&P 500 Market Index Application: Rolling window (II)

In this subsection we apply the model to analyze S&P 500 index returns on periods 2013-2018 using a rolling window.

Table 9: S&P 500 estimation results for the period 2016-2018.

Parameters	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
μ	0.0598**	0.0301	0.0470
	(0.0297)	(0.0304)	(0.0301)
ω	0.0394***	0.0370***	0.0344***
	(0.0057)	(0.0054)	(0.0051)
α	0.2146***	0.0507***	0.0581***
	(0.0213)	(0.0143)	(0.0171)
β	0.7382***	0.7634***	0.7701***
	(0.0297)	(0.0284)	(0.0288)
λ_1	0.0424	0.0319	-0.0749
	(0.0573)	(0.0544)	(0.0525)
γ	-	0.2556***	0.2527***
		(0.0298)	(0.0398)
λ_2	-	-	0.1914***
			(0.0483)
AIC	**1575.468**	**1557.305**	**1553.541**

Note: *** — $p < 0.01$, ** — $p < 0.05$, *– $p < 0.1$; st.errors in parentheses.

Asymmetry

Variance

Risk premium

The data exhibits the leverage effect both in volatility and return equations

The GARCH-M-GJR-LEV model should be applied

Here we present the results for the last 3-year sample of the observed period.

Insignificant estimate of λ_1

- GARCH-M was not able to capture the risk premium at all

Significant estimate of λ_2

- Only the GARCH-M-GJR-LEV model has identified the risk premium
- Estimate of λ_1 is statistically insignificant

1 ***Risk premium reacts only to 'bad' volatility periods, while 'bull' market volatility fluctuations do not increase the risk premium.***

- Pattern demonstrates the **irrationality of investors**
- The evidence **is consistent** with Kahneman and Tversky (1979) and Zhang (2006), Black (1976), Nelson (1991).

2 ***Without applying the GARCH-M-GJR-LEV model one may misidentify the presence of risk premium in returns.***

S&P 500 Market Index Application: Rolling window (III)

In this subsection we apply the model to analyze S&P 500 index returns on periods 2013-2018 using a rolling window.

Table 10: S&P 500 estimation results for the period 2015-2017.

Parameters	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
μ	0.0004	0.0011	0.0065
	(0.0313)	(0.0312)	(0.0164)
ω	0.0464***	0.0492***	0.0497***
	(0.0068)	(0.0086)	(0.0085)
α	0.2401***	0.0543***	0.0589***
	(0.0218)	(0.0134)	(0.0125)
β	0.6996***	0.7107***	0.7003***
	(0.0329)	(0.0403)	(0.0398)
λ_1	0.1664**	0.0892	0.0134
	(0.0650)	(0.0594)	(0.0324)
γ	-	0.3065***	0.3098***
		(0.0352)	(0.0351)
λ_2	-	-	0.1792***
			(0.0486)
AIC	**1553.335**	**1531.684**	**1523.906**

Note: *** — $p < 0.01$, ** — $p < 0.05$, *– $p < 0.1$; st.errors in parentheses.

Table 11: S&P 500 estimation results for the period 2014-2016.

Parameters	GARCH-M	GARCH-M-GJR	GARCH-M-GJR-LEV
μ	0.0004	0.0004	0.0004
	(0.0422)	(0.0397)	(0.0419)
ω	0.0691***	0.0703***	0.0679***
	(0.0123)	(0.0137)	(0.0139)
α	0.2499***	0.0233	0.0058
	(0.0262)	(0.0152)	(0.0174)
β	0.6874***	0.6996***	0.7172***
	(0.0399)	(0.0454)	(0.0461)
λ_1	0.1435**	0.0578	-0.0049
	(0.0656)	(0.0577)	(0.0635)
γ	-	0.3630***	0.3711***
		(0.0429)	(0.0402)
λ_2	-	-	0.1826**
			(0.0747)
AIC	**1753.369**	**1720.883**	**1715.978**

Note: *** — $p < 0.01$, ** — $p < 0.05$, *– $p < 0.1$; st.errors in parentheses.

All the three periods demonstrate the **same pattern**

Statistical evidence of asymmetric risk premium and volatility responses

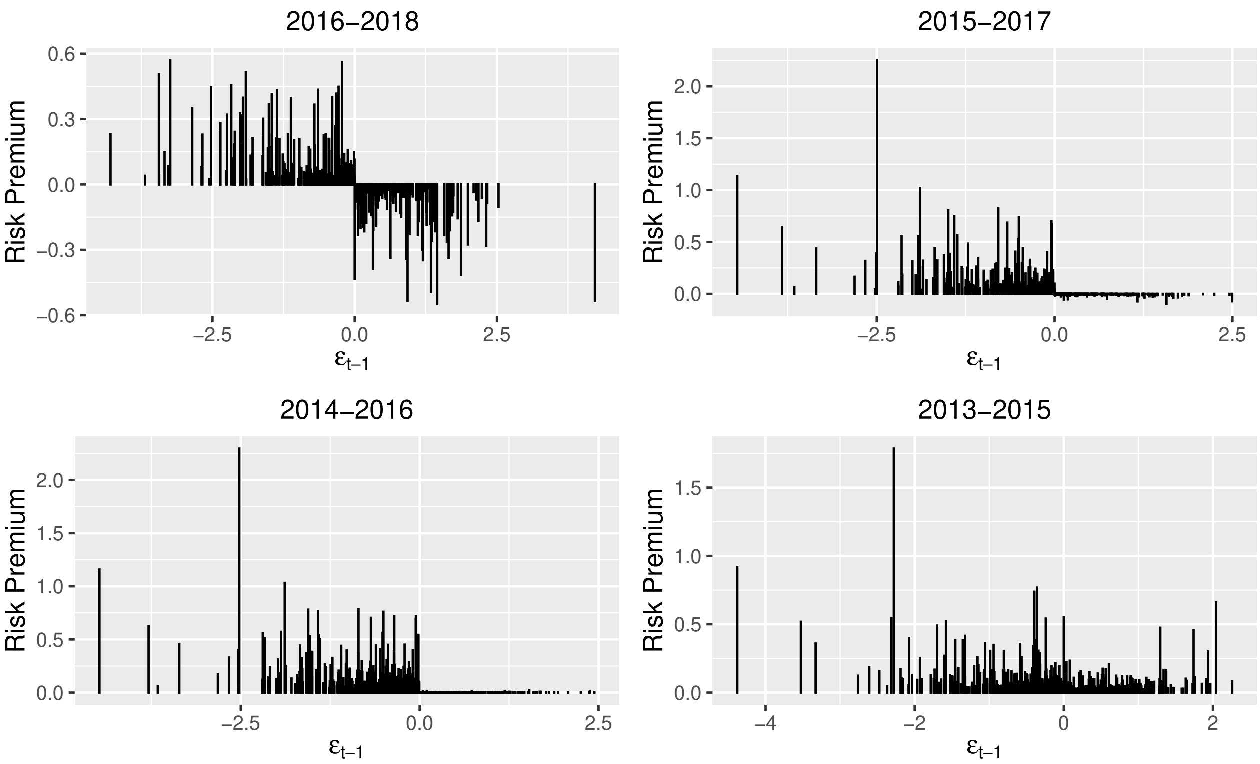
- Statistically **significant estimates** of λ_2, and γ
- GARCH-M-GJR does **not capture** the risk premium
- The pattern is **consistent over all periods**

Investors demand a risk premium only during 'bad' volatility periods.

- Pattern demonstrates the **irrationality of investors**
- The evidence **supports the volatility differentiation hypothesis** of (Bollerslev, 2022)

S&P 500 Market Index Application: Rolling window (IV)

All periods demonstrate an asymmetric relationship between the risk premium and volatility changes, based on the sign of shocks that cause volatility rises.



Two different patterns

2016-2018 | All other periods

The difference is explained only by the sign of the λ_1 estimate.

2016-2018 ⟫ λ_1 is negative and high

- *Investors demand a **risk premium only during "bad"** volatility periods, while **"good"** volatility may even **produce a discount**.*

Other ⟫ λ_1 is positive

- ***Risk premium rises during both** "good" and "bad" volatility periods, while **negative shocks** in returns **influence** the premium **more sharply***

Conclusion

*Since the data exhibits a leverage effect in risk premium, the proposed **GARCH-M-GJR-LEV** model **should be used**.*

References

Berkes, I. and L. Horvath (2004). The efficiency of the estimators of the parameters in garch processes. *The Annals of Statistics* 32, 633–655.

Black, F. (1976). Studies of stock price volatility changes. *Proceedings of the 1976 Meeting of the Business and Economic Statistics Section, American Statistical Association*, 177–181.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics* 31, 307–327.

Bollerslev, T. (2022). Realized semi(co)variation: Signs that all volatilities are not created equal. *Journal of Financial Econometrics* 20, 219–252.

Bollerslev, T., J. Litvinova, and G. Tauchen (2006). Leverage and volatility feedback effects in high-frequency data. *Journal of Financial Econometrics* 4, 353–384.

Christie, A. (1982). The stochastic behavior of common stock variances value, leverage and interest rate effects. *Journal of Financial Economics* 10, 407–432.

Engle, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of united kingdom inflation. *Econometrica* 50, 987–1007.

Engle, R. F., D. M. Lilien, and R. P. Robins (1987). Estimating time varying risk premia in the term structure: The arch-m model. *Econometrica* 55, 391–407.

Glosten, L. R., R. Jagannathan, and D. E. Runkle (1993). On the relation between the expected value and the volatility of the nominal excess return on stocks. *The Journal of Finance* 48, 1779–1801.

Hong, S. Y. and O. Linton (2020). Nonparametric estimation of infinite order regression and its application to the risk-return tradeoff. *Journal of Econometrics* 219, 389–424.

Kahneman, D. and A. Tversky (1979). Prospect theory: An analysis of decision under risk. *Econometrica* 47, 263–292.

Markowitz, H. (1952). Portfolio selection. *The Journal of Finance* 7, 77–91.

Miralles-Marcelo, J. L., J. L. Miralles-Quiros, and M. del Mar Miralles-Quiros (2013). Multivariate garch models and risk minimizing portfolios: The importance of medium and small firms. *The Spanish Review of Financial Economics* 11, 29–38.

Nelson, D. B. (1991). Conditional heteroskedasticity in asset returns: A new approach. *Econometrica* 59, 347–370.

Rossi, A. G. and A. Timmermann (2015). Modeling covariance risk in merton's icapm. *The Review of Financial Studies* 28, 1428–1461.

Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *The Journal of Finance* 19, 425–442.

Zhang, X. F. (2006). Information uncertainty and stock returns. *The Journal of Finance* 61, 105–137.

NATIONAL RESEARCH
UNIVERSITY

Phone.: +7 (925)-079-13-90

Email: ytrifonov@hse.ru