Comparison of GARCH and HAR-RV models for realized volatility of Bitcoin and E-mini S&P 500

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Research Goal

This work investigates the applicability of GARCH models with various error distribution specifications, as well as HAR models and their various specifications in the tasks of forecasting realized volatility one-step ahead in a rolling window of 399 values in increments of 5 values.

For each data series, 810 GARCH models are considered, as well as 46312 HAR-RV models, and the same number of models for logarithmic and root HAR-specifications.

Thus, a total of 139746 models are considered for each data series in the work. With the help of the MCS test, the best models are selected according to the accuracy of the one-day ahead forecast.

Data

As indicators of the cryptocurrency market and the stock market, the following exchange assets were selected for analysis:

- Bitcoin is the cryptocurrency with the largest capitalization, in fact the most representative cryptocurrency, with the greatest history. Traded 24/7.
- The E-mini S&P 500 is a futures contract traded on the Chicago Mercantile Exchange (CME), representing 20% of the value of a standard futures contract for the S&P 500 index. Trading is conducted from 6:00 a.m. on Sunday to 5:00 a.m. on Friday (Chicago Stock Exchange time) with a daily break from 5:00 a.m. to 6:00 a.m.

The data for the empirical analysis are closing 5-minutes prices of Bitcoin and E-mini S&P 500 futures. They have been obtained from finam.ru and they refer to the period from 1/1/2018 to 29/12/2021.

Bitcoin and E-mini S&P 500 Futures Returns

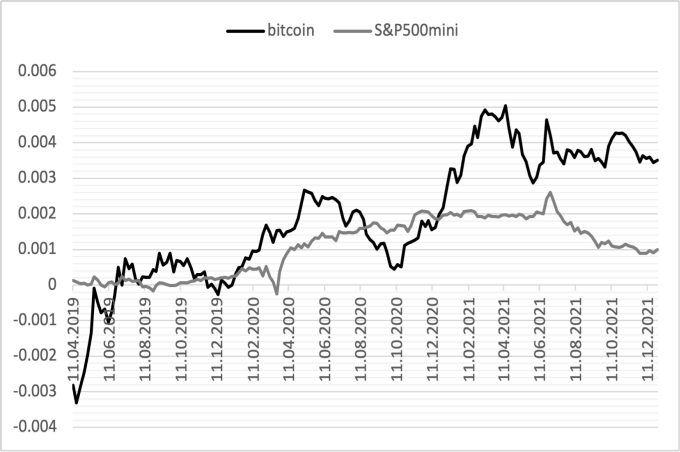

Figure 1: Average one-day returns of assets calculated in a rolling windows with a width of 399 days

Variances of Bitcoin and E-mini S&P 500 futures

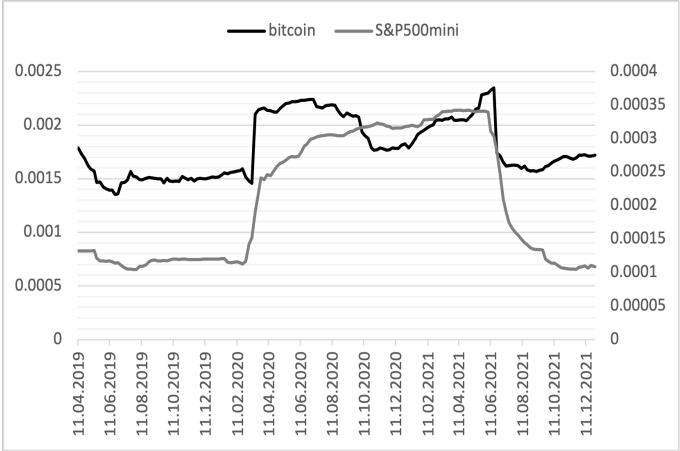

Figure 2: Sample variances of one-day returns calculated in a rolling windows with a width of 399 days. (The left scale is Bitcoin, the right scale is E-mini S&P 500 futures.)

Realized Volatility

The 5-minute realized volatility is used as the realized volatility. The choice of such time intervals is due to a study by Andersen and Bollerslev (1998).

The realized volatility per day on day t represented as:

$$RV_{t,j} = \sqrt{\sum_{j=1}^{288} r_{t,j}^2},$$

where $r_{t,j} = \log(p_{t,j}) - \log(p_{t,j-1})$ are the returns, $p_{t,j}$ is the price of an asset on day t at the end of an intraday interval j of length 5 minutes, with the total number of such intervals for one day equal to 288.

Realized Volatility

To obtain comparable results in case of gaps in the data, the realized volatility is calculated as follows:

- if the data is available for less than 5 hours in a day, then the corresponding day is removed from the sample;
- if observations are missing at the beginning and/or the end of the day, the realized volatility is calculated from the available K 5-minute intervals, and then reduced to daily data by scaling

$$RV_t = \sqrt{\frac{288}{K}\sum_{j=1}^{K} r_{t,j}^2};$$

- if data is missing within a day, for example, between the moments j_1 and j_2, the corresponding value is imputed by the square of the return for the missed period.

Realized volatility

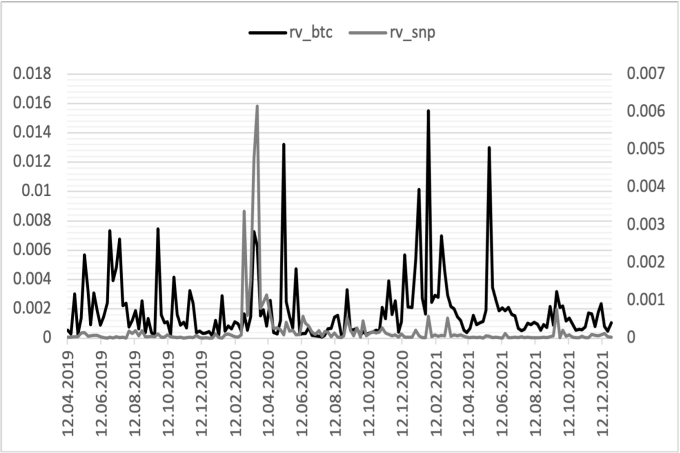

Figure 3: Realized volatility of Bitcoin (black line) and E-mini S&P 500 futures. (The left scale is bitcoin, the right scale is E-mini S&P 500 futures.)

Methodology

The models are compared according to the accuracy of the one-day ahead volatility forecast. Each model is evaluated in a rolling window with a width of 399 days and a forecast is made for one-day ahead. To reduce the calculation time, the window shift step is selected for 5 days.

To compare models by forecast accuracy, the MCS (Model Confidence Set) test introduced in (Hansen et al., 2011) is used. This test allows you to take into account the imperfection of the data and, if available, not one, but several models are selected that are equally better than others.

Comparison of Models

Three different loss functions are used as error metrics for comparison:

$$MSE = (RV_k - h_k)^2, \tag{1}$$

$$MAE = |RV_k - h_k|, \tag{2}$$

$$MAPE = \left| \frac{RV_k - h_k}{RV_k} \right| \times 100\%, \tag{3}$$

where RV_k is realized volatility on day k, h_k is forecast of realized volatility on day k.

The test is conducted at a significance level of 0.01.

GARCH Models I

All GARCH models are considered with an AR(1) part for returns:

$$r_t = \sum_{i=1}^{p} \phi_i r_{t-i} + \varepsilon_t,$$

where r_t is the return on day t, ϕ is the parameter.
The following ten GARCH models participated in the comparison:

1. Standard GARCH(p_1,q_1) models – GARCH(p_1,q_1) (Bollerslev (1986)):

$$\varepsilon_t = \sigma_t \xi_t$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q_1} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p_1} \beta_j \sigma_{t-j}^2,$$

where σ_t^2 is the conditional variance at time t, r_t is the return at time t, $\phi, \theta, \alpha, \beta$ are the parameters.

GARCH Models II

2. Exponential GARCH model – EGARCH (Nelson (1991)):

$$\ln(\sigma_t^2) = \alpha_0 + \sum_{i=1}^{q_1} [\alpha_i \varepsilon_{t-i} + \gamma_i(|\varepsilon_{t-i}| - E|\varepsilon_{t-i}|)] + \sum_{j=1}^{p_1} \beta_j \ln(\sigma_{t-j}^2).$$

3. Threshold GJR-GARCH (Glosten et al. (1993)) :

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q_1} (\alpha_i + \gamma I_{t-i}^-)\varepsilon_{t-i}^2 + \sum_{j=1}^{p_1} \beta_j \sigma_{t-j}^2, \quad I_{t-i}^- = \begin{cases} 1, & \varepsilon_t < 0 \\ 0, & \varepsilon_t \geq 0 \end{cases}.$$

4. Assymetric Power ARCH – APARCH (Ding et al. (1993)):

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q_1} (\alpha_i(|\varepsilon_{t-i}| - \gamma_i \varepsilon_{t-i})^2 + \sum_{j=1}^{p_1} \beta_j \sigma_{t-j}^2,$$

where $-1 < \gamma < 1$ is an indicator of accounting for asymmetry.

Error distribution I

1. Standard normal distribution with density function

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

2. Skewed normal distribution (O'Hagan and Leonard (1976)).
   The density function of a skewed normal distribution with parameter α:

$$f(x) = 2\phi(x)\Phi(\alpha x),$$

   where $\phi(x)$ is the probability density function of the standard normal distribution, $\Phi(x)$ is the cumulative distribution function. The variance is equal to $1 - \frac{2}{\pi}\frac{\alpha^2}{1+\alpha^2}$.

Error distribution II

3. Normal-inverse Gaussian distribution.

$$f(x,\theta) = \frac{\alpha\delta}{\pi} e^{\delta\gamma+\beta(x-\mu)} \frac{K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}},$$

where $x \in R, \alpha > 0, \beta \in (-\alpha,\alpha), \delta \in (0,\infty), \gamma = \sqrt{\alpha^2-\beta^2}, K_1(w) = \frac{1}{2}\int\limits_0^\infty e^{-\frac{(w(t+t^{-1}))}{2}} dt$ is a modified Bessel function of the third kind with index 1. The variance of this distribution is equal to $\alpha^2\delta(\alpha^2-\beta^2)^{-3/2}$. To normalize the variance by 1, you can take $\delta = (\alpha^{-2}-\beta^2)^{3/2}$.

4. Student's t-distribution with v degrees of freedom.

$$f(x) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{\beta v\pi}\Gamma\left(\frac{v}{2}\right)} \left(1+\frac{(x-\alpha)^2}{\beta v}\right)^{-\frac{v+1}{2}}.$$

To normalize the variance by 1, we assume $\beta = \frac{v-2}{2}$.

HAR Models I

The paper evaluates the models presented below, as well as their logarithmic and root specifications, which in total allows us to sort through 138936 different HAR models.

1. HAR(w,m) (Corsi, 2003):

$$RV_{t+1}^{d} = \beta_0 + \beta_1 RV_t^d + \beta_2 RV_t^w + \beta_3 RV_t^m + \varepsilon_{t+1}.$$

2. HARJ(w,m) with order BPV(w_1,m_1) – HAR with Jump (Andersen et al., 2007):

$$RV_{t+1}^{d} = \beta_0 + \beta_1 RV_t^d + \beta_2 RV_t^w + \beta_3 RV_t^m + \beta_4 J_t^d + \beta_5 J_t^{w_1} + \beta_6 J_t^{m_1} + \varepsilon_{t+1}.$$

3. HARCJ(w,m) with order BPV(w_1,m_1) – HAR with Jump and Continuous sample path (Andersen et al., 2007):

$$RV_{t+1}^{d} = \beta_0 + \beta_1 BPV_t^d + \beta_2 BPV_t^w + \beta_3 BPV_t^m + \beta_4 J_t^d + \beta_5 J_t^{w_1} + \beta_6 J_t^{m_1} + \varepsilon_{t+1}.$$

4. HARQ(w,m) with order RQ(w_1,m_1) – HAR with Realized Quarticity (Bollerslev et al., 2016):

$$RV_{t+1}^{d} = \beta_0 + \beta_1 RV_t^d + \beta_2 RV_t^w + \beta_3 RV_t^m + \beta_4 RQ_t^d + \beta_5 RQ_t^{w_1} + \beta_6 RQ_t^{m_1} + \varepsilon_{t+1}.$$

HAR Models II

5. HARQJ(w,m) with order BPV(w_1,m_1) and RQ(w_2,m_2) – HAR with Realized Quarticity and Jump (Bollerslev et al., 2016):

$$RV_{t+1}^d = \beta_0 + \beta_1 RV_t^d + \beta_2 RV_t^w + \beta_3 RV_t^m + \beta_4 J_t^d + \beta_5 J_t^{w_1} + \beta_6 J_t^{m_1} +$$

$$+\beta_7 RQ_t^d + \beta_8 RQ_t^{w_2} + \beta_9 RQ_t^{m_2} + \varepsilon_{t+1}.$$

6. CHAR(w,m) – HAR with Continuous sample path (Andersen et al., 2007):

$$RV_{t+1}^d = \beta_0 + \beta_1 J_t^d + \beta_2 J_t^w + \beta_3 J_t^m + \varepsilon_{t+1}.$$

7. CHARQ(w,m) with order RQ(w_1,m_1) – HAR with Realized Quarticity and Continuous sample path (Bollerslev et al., 2016):

$$RV_{t+1}^d = \beta_0 + \beta_1 J_t^d + \beta_2 J_t^w + \beta_3 J_t^m + \beta_4 RQ_t^d + \beta_5 RQ_t^{w_1} + \beta_6 RQ^{m_1} + \varepsilon_{t+1}.$$

In the logarithmic and root specifications, instead of RV_t^d and other components (RQ_t, BPV_t, J_t), $\ln(RV_t^d)$ and $\sqrt{RV_t^d}$ are taken, respectively.

Types of Realized Volatility in HAR-models

The description of the HAR models below uses:

- RV_t^d – realized variance on day t, the square of the realized volatility;
- $RV_t^w = \frac{1}{w}\sum_{j=0}^{w-1} RV_t^d$ – average realized volatility for the weekly period;
- $RV_t^m = \frac{1}{m}\sum_{j=0}^{m-1} RV_t^d$ – average realized volatility for the monthly periods.

The standard HAR-RV model (Corsi, 2009) uses $w = 5, m = 21$ for the stock index. Since cryptocurrency trading differs from conventional exchange trading, all pairs (m,w) are used in this work, where $4 \leqslant w \leqslant 7$ and $21 \leqslant m \leqslant 27$.

Results for E-mini S&P 500 futures: GARCH-models

Metric	Best models	MAE	MAPE, %
MAE	csGARCH(1,0)-snorm	0.01277	33.45%
MSE	csGARCH(2,2)-snorm	0.01415	46.11%
MAPE	csGARCH(1,0)-ged	0.01277	33.45%
Benchmark	GARCH(1,1)-norm	0.01521	49.51%

Table 1: The best GARCH models for the realized volatility of Bitcoin

Metric	Best models	MAE	MAPE, %
MAE	NAGARCH(1,2)-std	00.004029	54.80%
MSE	NAGARCH(1,2)-jsu	0.004081	58.10%
MAPE	csGARCH(1,0)-snorm	0.004178	47.18%
Benchmark	GARCH(1,1)-norm	0.004623	61.97%

Table 2: The best GARCH models for the realized volatility of E-mini S&P500

Results for Bitcoin: HAR-models

RV	Metric	Best models	MAE	MAPE, %
HAR-RV	MAE	HARJ_RV(7,27)_BPV(7,26)	0.01376	45.92%
HAR-RV	MSE	HAR_RV(4,21)	0.01377	45.20%
HAR-RV	MAPE	HAR_RV(4,21)	0.01377	45.20%
HAR-RV	Benchmark	HAR_RV(5,21)	0.01399	45.72%
HAR-$\sqrt{RV}$	MAE	HARJ_RV(6,27)_BPV(7,24)	0.01193	33.46%
HAR-$\sqrt{RV}$	MSE	HARJ_RV(6,27)_BPV(5,23)	0.01200	33.34%
HAR-$\sqrt{RV}$	MAPE	HARJ_RV(6,27)_BPV(5,23)	0.01200	33.34%
HAR-$\sqrt{RV}$	Benchmark	HAR_RV(5,21)	0.01227	34.00%
HAR-ln(RV)	MAE	CHARQ_RV(5,27)_BPV(4,26)_RQ(5,24)	0.01146	29.74%
HAR-ln(RV)	MSE	CHARQ_RV(7,27)_BPV(6,24)_RQ(7,22)	0.01155	30.25%
HAR-ln(RV)	MAPE	HARQJ_RV(5,27)_BPV(4,21)_RQ(6,24)	0.01165	29.51%
HAR-ln(RV)	Benchmark	HAR_RV(5,21)	0.01192	30.51%

Table 3: The best HAR-RV models for the realized volatility of Bitcoin

Results for E-mini S&P 500 futures: HAR Models

RV	Metric	Best models	MAE	MAPE, %
HAR-RV	MAE	HARJ_RV(4,25)_BPV(7,22)	0.003819	50.82%
HAR-RV	MSE	HARJ_RV(4,25)_BPV(5,26)	0.003823	51.76%
HAR-RV	MAPE	HARJ_RV(5,26)_BPV(7,22)	0.003884	47.69%
HAR-RV	Benchmark	HAR_RV(5,21)	0.003883	52.13%
HAR-$\sqrt{RV}$	MAE	HARQJ_RV(7,24)_BPV(4,25)_RQ(7,26)	0.003146	40.44%
HAR-$\sqrt{RV}$	MSE	HARQJ_RV(7,23)_BPV(5,24)_RQ(5,25)	0.003186	41.23%
HAR-$\sqrt{RV}$	MAPE	CHARQ_RV(5,24)_BPV(7,27)_RQ(4,22)	0.003328	39.59%
HAR-$\sqrt{RV}$	Benchmark	HAR_RV(5,21)	0.003464	41.40%
HAR-ln(RV)	MAE	HARCJ_RV(5,27)_BPV(5,24)	0.003211	36.37%
HAR-ln(RV)	MSE	HARCJ_RV(5,27)_BPV(4,24)	0.003216	36.38%
HAR-ln(RV)	MAPE	CHAR_RV(5,24)	0.003377	36.12%
HAR-ln(RV)	Benchmark	HAR_RV(5,21)	0.003452	37.19%

Table 4: The best HAR-RV models for the realized volatility of the E-mini S&P 500 futures

Realized Volatility of Bitcoin: forecast

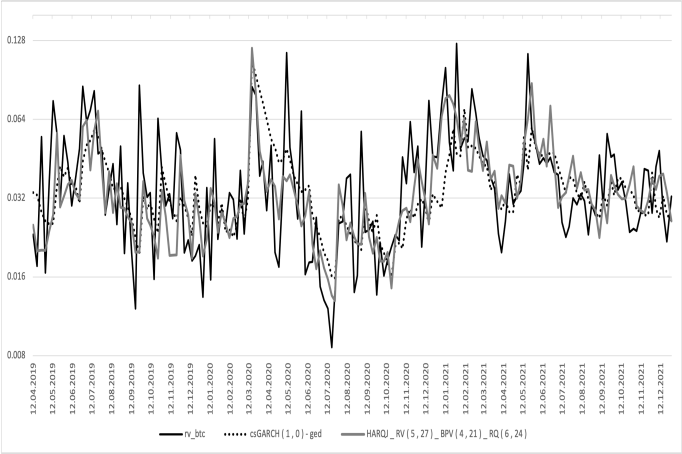

Figure 4: The black solid line is the realized volatility of Bitcoin, the dotted line and the gray line are its forecasts for the best GARCH and HAR models

Scatter Plot for Bitcoin

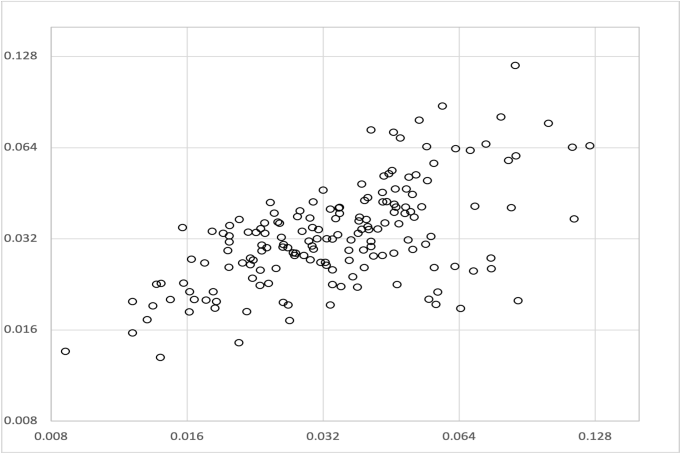

Figure 5: Scatter plot. Horizontally - the realized volatility of Bitcoin, vertically its forecast according to the HARQJ-RV(5,27)-BPV(4,21)-RQ(6,24) model

Realized Volatility of E-mini S&P 500: forecast

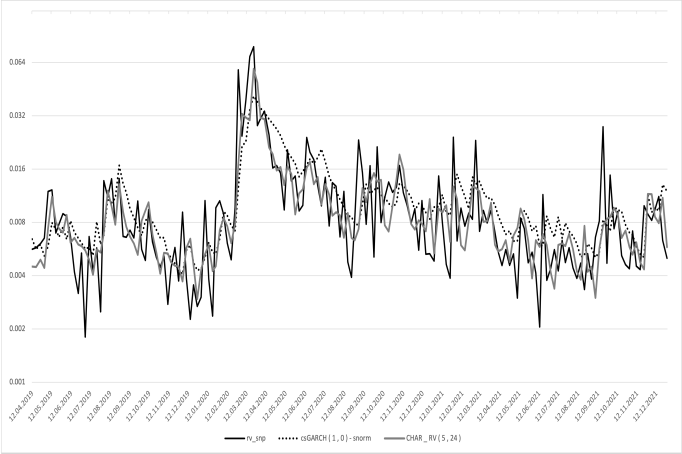

Figure 6: The black solid line is the realized volatility of the E-mini S&P 500 futures, dotted and gray - its forecasts for the best GARCH and HAR models

Scatter Plot for E-mini S&P 500 Futures

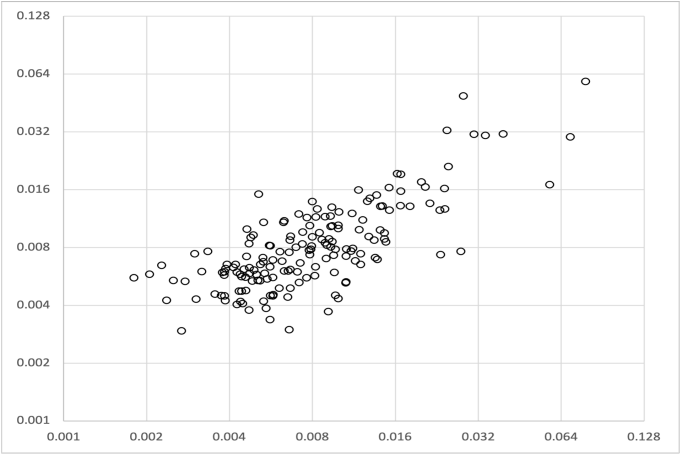

Figure 7: Scatter plot. Horizontally — the realized volatility of the E-mini S&P 500 futures, vertically its forecast according to the CHAR-RV model(5,24)

Conclusion

1. GARCH models are inferior to HAR models in the accuracy of the realized volatility forecast as Bitcoin and E-mini S&P 500 futures.
2. For the best HAR models, the relative accuracy of the realized volatility forecast for Bitcoin is higher than for the E-mini S&P 500 futures. Perhaps, this indicates a greater segmentation (heterogeneity) of the Bitcoin market. The smallest average relative errors (MAPE) that were achieved were 29.51% and 36.12% for Bitcoin and E-mini S&P 500 futures, respectively.
3. Among all the selected models of the GARCH family, most have a skewed normal distribution and a generalized error distribution. Such models allow us to take into account both long-term and short-term fluctuations of volatility.
4. For both time series studied, the best results were shown by the specification models HAR-ln(RV), which is consistent with the lognormal nature of the realized volatility.
5. For both time series, among the best HAR models, models were selected that take into account not only the heterogeneity of the market, but also the continuous component and jumps. Such models slightly exceed the standard HAR-RV(5,21) model in terms of forecast accuracy.

Thank you for your attention!

Application 1. GARCH Models I

1. Assymetric Power ARCH – APARCH (Ding et al. (1993)):

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q_1} (\alpha_i(|\varepsilon_{t-i}| - \gamma_i \varepsilon_{t-i})^2 + \sum_{j=1}^{p_1} \beta_j \sigma_{t-j}^2,$$

   where $-1 < \gamma < 1$ is an indicator of accounting for asymmetry.

2. Component sGARCH – csGARCH (Lee and Engle (1999)):

$$\sigma_t^2 = y_t + \sum_{i=1}^{q_1} \alpha_i(\varepsilon_{t-i}^2 - q_{t-i}) + \sum_{j=1}^{p_1} \beta_j(\sigma_{t-j}^2 - q_{t-j})$$

$$y_t = w + \rho y_{t-1} + \phi(\varepsilon_{t-1}^2 + \sigma_{t-1}^2),$$

   where q_t is the constant component of the conditional variance. The difference between the conditional variance and its trend $(\sigma_{t-j}^2 - q_{t-j})$ is a temporary (transitive) component of the conditional variance. Stationarity condition:

Application 1. GARCH Models II

$$\sum_{i=1}^{q_1} \alpha_i + \sum_{j=1}^{p_1} \beta_j < 1, \quad \rho < 1$$

.

3. ALLGARCH - HGARCH (Hentschel (1995)):

$$\sigma_t^2 = w + \sum_{i=1}^{q_1} \alpha_i \sigma_{t-i}^2 (|\varepsilon_{t-i} - \tau_{2i}| - \tau_{1i}(\varepsilon_{t-i} - \tau_{2i}))^2 + \sum_{j=1}^{p_1} \beta_j \sigma_{t-j}^2.$$

4. Asymmetric Absolute Value GARCH – AVGARCH (Taylor (1986) and Schwert (1990)):

$$\sigma_t = w + \sum_{i=1}^{q_1} \alpha_i (|\varepsilon_{t-i} - \tau_{2i}| - \tau_{1i}(\varepsilon_{t-i} - \tau_{2i})) + \sum_{j=1}^{p_1} \beta_j \sigma_{t-j}.$$

Application 1. GARCH Models III

5. Threshold GARCH – TGARCH (Zakoian (1994)):

$$\sigma_t = \alpha_0 + \sum_{i=1}^{q_1} (\alpha_i^+ \varepsilon_{t-i}^+ + \alpha_i^- \varepsilon_{t-i}^-) + \sum_{j=1}^{p_1} \beta_j \sigma_{t-j}, \quad \begin{cases} \varepsilon^+ = max(0,\varepsilon) \\ \varepsilon^- = min(0, -\varepsilon) \end{cases} .$$

6. Nonlinear ARCH – NARCH (Higgins et al. (1992)):

$$\sigma_t^\delta = w + \sum_{i=1}^{q_1} \alpha_i |\varepsilon_{t-i}|^\delta + \sum_{j=1}^{p_1} \beta_j \sigma_{t-j}^\delta.$$

7. Nonlinear Asymmetric GARCH – NAGARCH (Engle and Ng (1993)):

$$\sigma_t^2 = w + \sum_{i=1}^{q_1} \alpha_i (\varepsilon_{t-i} + \gamma_i \sigma_{t-i})^2 + \sum_{j=1}^{p_1} \beta_j \sigma_{t-j}^2.$$

Generalizations of HAR Models

Generalizations of HAR models use:

1. Realised bipower variation (Barndorff-Nielsen, Shepard, 2004):

$$BPV_t = \sum_{j=1}^{N-1} r_{j,t} r_{j+1,t}.$$

2. Realized quarticity (Corsi et. all, 2005):

$$RQ_t = \frac{N}{3} \sum_{j=1}^{N} r_{j,t}^4.$$

3. Jump (Barndorff-Nielsen, Shepard, 2004):

$$J_t = max(RV_t^d - BPV_t, 0).$$

4. Continuous sample path (Andersen et al., 2007):

$$C_t = RV_t^d - J_t.$$