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On Noncommutative Vieta Theorem in Geometric Algebras

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- We discuss a generalization of Vieta theorem (Vieta's formulas) to the case of Clifford geometric algebras.
- We compare the generalized Vieta's formulas with the ordinary Vieta's formulas for characteristic polynomial containing eigenvalues.
- We discuss Gelfand - Retakh noncommutative Vieta theorem and use it for the case of geometric algebras of small dimensions.
- The results can be used in symbolic computation and various applications of geometric algebras in computer science, computer graphics, computer vision, physics, and engineering.

Geometric Algebras

Let us consider the real (Clifford) geometric algebra $\mathcal{G}_{p,q}$, $n = p+q \geq 1$ [12,5,14] with the generators e_a, $a = 1, 2, \ldots, n$ and the identity element $e \equiv 1$. The generators satisfy

$$e_a e_b + e_b e_a = 2\eta_{ab} e, \qquad a, b = 1, 2, \ldots, n,$$

where $\eta = (\eta_{ab})$ is the diagonal matrix with its first p entries equal to 1 and the last q entries equal to -1 on the diagonal. The grade involution and reversion of an arbitrary element (a multivector) $U \in \mathcal{G}_{p,q}$ are denoted by

$$\widehat{U} = \sum_{k=0}^{n} (-1)^k \langle U \rangle_k, \qquad \widetilde{U} = \sum_{k=0}^{n} (-1)^{\frac{k(k-1)}{2}} \langle U \rangle_k,$$

where $\langle U \rangle_k$ is the projection of U onto the subspace $\mathcal{G}^k_{p,q}$ of grade $k = 0, 1, \ldots, n$.

Characteristic polynomial in Geometric algebras

Let us consider the following faithful representation (isomorphism) of the complexified geometric algebra $\mathbb{C} \otimes \mathcal{G}_{p,q}$, $n = p + q$

$$\beta : \mathbb{C} \otimes \mathcal{G}_{p,q} \to M_{p,q} := \begin{cases} \mathrm{Mat}(2^{\frac{n}{2}}, \mathbb{C}) & \text{if } n \text{ is even,} \\ \mathrm{Mat}(2^{\frac{n-1}{2}}, \mathbb{C}) \oplus \mathrm{Mat}(2^{\frac{n-1}{2}}, \mathbb{C}) & \text{if } n \text{ is odd.} \end{cases} \tag{1}$$

The real geometric algebra $\mathcal{G}_{p,q}$ is isomorphic to some subalgebra of $M_{p,q}$, because $\mathcal{G}_{p,q} \subset \mathbb{C} \otimes \mathcal{G}_{p,q}$ and we can consider the representation of not minimal dimension

$$\beta : \mathcal{G}_{p,q} \to \beta(\mathcal{G}_{p,q}) \subset M_{p,q}.$$

We can introduce (see [17]) the notion of determinant

$$\mathrm{Det}(U) := \det(\beta(U)) \in \mathbb{R}, \qquad U \in \mathcal{G}_{p,q}$$

and the notion of characteristic polynomial

$$\varphi_U(\lambda) := \mathrm{Det}(\lambda e - U) = \lambda^N - C_{(1)}\lambda^{N-1} - \cdots - C_{(N-1)}\lambda - C_{(N)} \in \mathcal{G}^0_{p,q} \equiv \mathbb{R},$$
$$U \in \mathcal{G}_{p,q}, \qquad N = 2^{[\frac{n+1}{2}]}, \qquad C_{(k)} = C_{(k)}(U) \in \mathcal{G}^0_{p,q} \equiv \mathbb{R}, \quad k = 1, \ldots, N, \tag{2}$$

where $\mathcal{G}^0_{p,q}$ is a subspace of elements of grade 0, which we identify with scalars.

Ordinary Vieta's formulas

Let us denote the solutions of the characteristic equation $\varphi_U(\lambda) = 0$ (i.e. eigenvalues) by $\lambda_1, \ldots, \lambda_N$. By the Vieta's formulas from matrix theory, we know that

$$C_{(k)} = (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}, \qquad k = 1, \ldots, N,$$

in particular,

$$C_{(1)} = \lambda_1 + \cdots + \lambda_N = \mathrm{Tr}(U), \qquad \ldots, \qquad C_{(N)} = -\lambda_1 \cdots \lambda_N = -\mathrm{Det}(U),$$

where $\mathrm{Tr}(U) := \mathrm{tr}(\beta(U)) = N\langle U \rangle_0$ is the trace of U. The elements $C_{(k)}$, $k = 1, \ldots, N$ are elementary symmetrical polynomials in the variables $\lambda_1, \ldots, \lambda_N$.

The case n=1

In this case, the geometric algebra $\mathcal{G}_{p,q}$ is commutative and $N = 2$. We have

$$C_{(1)} = \lambda_1 + \lambda_2 \in \mathbb{R}, \qquad C_{(2)} = -\lambda_1\lambda_2 \in \mathbb{R}. \tag{3}$$

But also we have

$$C_{(1)} = U + \widehat{U} \in \mathcal{G}^0_{p,q} \equiv \mathbb{R}, \qquad C_{(2)} = -U\widehat{U} \in \mathcal{G}^0_{p,q} \equiv \mathbb{R}. \tag{4}$$

The elements $y_1 := U$ and $y_2 := \widehat{U}$ are not scalars (and are not equal to the eigenvalues λ_1 and λ_2), but they are solutions of the characteristic equation $\varphi_U(x) = 0$, $x = y_1, y_2$ by the Cayley – Hamilton theorem. Using

$$\lambda^2 - (U + \widehat{U})\lambda + U\widehat{U} = 0,$$

we get the explicit formulas for the eigenvalues

$$\lambda_{1,2} = \frac{1}{2}(U + \widehat{U} \pm \sqrt{(U + \widehat{U})^2 - 4U\widehat{U}}) = \frac{1}{2}(U + \widehat{U} \pm \sqrt{(U - \widehat{U})^2}) = \langle U\rangle_0 \pm \sqrt{(\langle U\rangle_1)^2}, \tag{5}$$

which do not coincide with the explicit formulas for $y_{1,2}$

$$y_{1,2} = \langle U\rangle_0 \pm \langle U\rangle_1, \tag{6}$$

because the scalar $\sqrt{(\langle U\rangle_1)^2}$ does not coincide with the vector (element of grade 1) $\langle U\rangle_1$. We see that the role of the roots $\lambda_{1,2}$ (which are complex scalars) of the characteristic equation is played by some combinations $y_{1,2}$ of involutions of elements (which are not scalars). In the case of degenerate eigenvalues, we have $\langle U\rangle_1 = 0$ and the coincidence $\lambda_{1,2} = y_{1,2} = U = \langle U\rangle_0$.

The case n=2

We have $N = 2$ and

$$C_{(1)} = \lambda_1 + \lambda_2 \in \mathbb{R}, \qquad C_{(2)} = -\lambda_1\lambda_2 \in \mathbb{R}. \tag{7}$$

But also we have

$$C_{(1)} = U + \widehat{\widetilde{U}} \in \mathcal{G}^0_{p,q} \equiv \mathbb{R}, \qquad C_{(2)} = -U\widehat{\widetilde{U}} \in \mathcal{G}^0_{p,q} \equiv \mathbb{R}. \tag{8}$$

Note that $U\widehat{\widetilde{U}} = \widehat{\widetilde{U}}U$ in the case $n = 2$. The elements $y_1 := U$ and $y_2 := \widehat{\widetilde{U}}$ are not scalars (and are not equal to the eigenvalues λ_1 and λ_2), but they are solutions of the characteristic equation $\varphi_U(x) = 0$, $x = y_1, y_2$ by the Cayley – Hamilton theorem. Using

$$\lambda^2 - (U + \widehat{\widetilde{U}})\lambda + U\widehat{\widetilde{U}} = 0,$$

we get the explicit formulas for the eigenvalues

$$\lambda_{1,2} = \frac{1}{2}(U + \widehat{\widetilde{U}} \pm \sqrt{(U + \widehat{\widetilde{U}})^2 - 4U\widehat{\widetilde{U}}}) = \frac{1}{2}(U + \widehat{\widetilde{U}} \pm \sqrt{(U - \widehat{\widetilde{U}})^2}) = \langle U\rangle_0 \pm \sqrt{(\langle U\rangle_1 + \langle U\rangle_2)^2}, \tag{9}$$

which do not coincide with the explicit formulas for $y_{1,2}$

$$y_{1,2} = \langle U\rangle_0 \pm (\langle U\rangle_1 + \langle U\rangle_2), \tag{10}$$

where the scalar $\sqrt{(\langle U\rangle_1 + \langle U\rangle_2)^2} = \sqrt{(\langle U\rangle_1)^2 + (\langle U\rangle_2)^2}$ is not equal to the expression $\langle U\rangle_1 + \langle U\rangle_2$. The role of the roots $\lambda_{1,2}$ (which are complex scalars) of the characteristic equation is played by some combinations $y_{1,2}$ of involutions of elements (which are not scalars).

In the case of degenerate eigenvalues, we have $\langle U\rangle_1 = \langle U\rangle_2 = 0$ and the coincidence $\lambda_{1,2} = y_{1,2} = U = \langle U\rangle_0$ in the case of two Jordan blocks; or $(\langle U\rangle_1)^2 = -(\langle U\rangle_2)^2 \neq 0$ and $\lambda_{1,2} = \langle U\rangle_0 \neq y_{1,2} = \langle U\rangle_0 \pm (\langle U\rangle_1 + \langle U\rangle_2)$ in the case of one Jordan block. For example, for $U = 5e + \frac{1}{2}(e_2 + e_{12})$, we have $\lambda_{1,2} = 5$ and $y_{1,2} = 5e \pm \frac{1}{2}(e_2 + e_{12})$ in the case $n = p = 2$, $q = 0$.

The case n=3

Let us consider the case $n = 3$. We have $N = 4$ and the formulas

$$C_{(1)} = U + \widehat{U} + \widetilde{U} + \widehat{\widetilde{U}}, \qquad C_{(2)} = -(U\widetilde{U} + U\widehat{U} + U\widehat{\widetilde{U}} + \widehat{U}\widehat{\widetilde{U}} + \widetilde{U}\widehat{\widetilde{U}} + \widehat{U}\widetilde{U}),$$
$$C_{(3)} = U\widehat{U}\widetilde{U} + U\widehat{U}\widehat{\widetilde{U}} + U\widetilde{U}\widehat{\widetilde{U}} + \widehat{U}\widetilde{U}\widehat{\widetilde{U}}, \qquad C_{(4)} = -U\widehat{U}\widetilde{U}\widehat{\widetilde{U}}. \tag{11}$$

These formulas look like the ordinary Vieta's formulas for eigenvalues:

$$C_{(1)} = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \qquad C_{(2)} = -(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4),$$
$$C_{(3)} = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4, \qquad C_{(4)} = -\lambda_1\lambda_2\lambda_3\lambda_4. \tag{12}$$

The elements

$$y_1 := U = \langle U\rangle_0 + \langle U\rangle_1 + \langle U\rangle_2 + \langle U\rangle_3, \qquad y_2 := \widetilde{U} = \langle U\rangle_0 + \langle U\rangle_1 - \langle U\rangle_2 - \langle U\rangle_3,$$
$$y_3 := \widehat{U} = \langle U\rangle_0 - \langle U\rangle_1 + \langle U\rangle_2 - \langle U\rangle_3, \qquad y_4 := \widehat{\widetilde{U}} = \langle U\rangle_0 - \langle U\rangle_1 - \langle U\rangle_2 + \langle U\rangle_3,$$

are not scalars (and are not equal to the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$), but they are solutions of the characteristic equation $\varphi_U(x) = 0$, $x = y_1, y_2, y_3, y_4$ by the Cayley – Hamilton theorem.

We call the formulas (4), (8), (11) and their analogues for the cases $n \geq 4$ *generalized Vieta's formulas in geometric algebra*. The formulas (4), (8), (11) were proved in [17] using recursive formulas for the characteristic polynomial coefficients following from the Faddeev – LeVerrier algorithm. In this work, we present an alternative proof of these formulas using the techniques of noncommutative symmetric functions.

Gelfand – Retakh theorem

Let us discuss the following Gelfand – Retakh theorem (known as the *noncommutative Vieta theorem* [10]).

Theorem 1. *If $\{x_1, \dots, x_N\}$ is an ordered generic set (i.e. Vandermonde quasideterminants v_k are defined and invertible for all $k = 1, \dots, N$) of solutions of the equation*

$$P_N(x) := x^N - a_1 x^{N-1} - \cdots - a_N = 0 \tag{13}$$

over a skew-field, then for $k = 1, 2, \dots, N$:

$$a_k = (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le N} y_{i_k} \cdots y_{i_1},$$

where

$$y_k = v_k x_k v_k^{-1}.$$

In [10], the definition of Vandermonde quasideterminants v_k is given (see also [8,9]). In this paper, we use another definition of the elements v_k from [7]:

$$v_k = P_{k-1}(x_k) = x_k^{k-1} - (y_{k-1} + \cdots + y_1) x_k^{k-2} + \cdots + (-1)^{k-1} y_{k-1} \cdots y_1. \tag{14}$$

In particular, we have

$$v_1 = 1, \qquad v_2 = x_2 - y_1, \qquad v_3 = x_3^2 - (y_2 + y_1) x_3 + y_2 y_1. \tag{15}$$

Remark 1. The condition $[v_k, x_k] = 0$ is equivalent to

$$[E_j, x_k] = 0, \qquad j = 1, \ldots, k-1,$$

where E_j, $j = 1, \ldots, k-1$ are noncommutative elementary symmetric polynomials in the variables $y_{k-1}, \ldots, y_1$:

$$E_1 = y_{k-1} + \cdots + y_1, \qquad E_{k-1} = y_{k-1} \cdots y_2 y_1.$$

For example, in the particular case $N = 4$, when all $[v_k, x_k] = 0$, $k = 1, \ldots, N$, we can take $y_k = x_k$, $k = 1, 2, 3, 4$, in the case

$$[x_2, x_1] = 0, \quad [x_3, x_2 x_1] = 0, \quad [x_3, x_2 + x_1] = 0, \quad [x_4, x_3 x_2 x_1] = 0,$$
$$[x_4, x_3 x_2 + x_3 x_1 + x_2 x_1] = 0, \quad [x_4, x_3 + x_2 + x_1] = 0.$$

We use this particular case below in $\mathcal{G}_{p,q}$ with $n = p + q = 3$.

Application of noncommutative Vieta theorem to Geometric algebras

Let us apply Theorem 1 to the particular case of the characteristic polynomial $\varphi_U(\lambda)$ in geometric algebra $\mathcal{G}_{p,q}$. The elements $a_k = C_{(k)} \in \mathbb{R}$, $k = 1, \ldots, N$ from (13) are scalars now. We need N solutions $x_1, x_2, \ldots x_N$ of the characterstic equation $\varphi_U(x) = 0$. By the Cayley – Hamilton theorem, we can take $x_1 = U$:

$$\varphi_U(U) = 0. \tag{19}$$

Theorem 2. *We have*

$$\varphi_U(\lambda) = \varphi_{\widehat{U}}(\lambda) = \varphi_{\widetilde{U}}(\lambda) = \varphi_{\widetilde{\widehat{U}}}(\lambda), \tag{20}$$

$$\varphi_U(\widetilde{U}) = \varphi_U(\widehat{U}) = \varphi_U(\widehat{\widetilde{U}}) = 0. \tag{21}$$

The case $n = 1$

In this case, the geometric algebra is commutative. We can take $y_1 = x_1 = U$ in Theorem 1 by the Cayley – Hamilton theorem. The element $x_2 = \widehat{U}$ satisfies the characteristic equation (see Theorem 2). We have $v_2 = x_2 - x_1 = \widehat{U} - U = -2\langle U\rangle_1$. If $\langle U\rangle_1 \neq 0$, then $y_2 = x_2 = \widehat{U}$ and we obtain the formulas (4).

The case $n = 2$

We can take $y_1 = x_1 = U$ in Theorem 1 by the Cayley – Hamilton theorem. The element $x_2 = \widetilde{U}$ satisfies the characteristic equation (see Theorem 2). We have $v_2 = x_2 - x_1 = \widetilde{U} - U = -2\langle U\rangle_2$. If $\langle U\rangle_2 = 0$, then we can use the formulas from the case $n = 1$. If $\langle U\rangle_2 \neq 0$, then $v_2 = \lambda e_{12}$, $\lambda \neq 0$ is invertible and $y_2 = v_2\widetilde{U}v_2^{-1} = \widetilde{\widehat{U}}$ because the pseudoscalar e_{12} commutes with all even elements and anticommutes with all odd elements. We get the formulas (8).

The case $n = 3$

We can take $y_1 = x_1 = U$ in Theorem 1 by the Cayley – Hamilton theorem.

Let us consider $x_2 = \widehat{\widetilde{U}}$. We have $v_2 = x_2 - x_1 = \widehat{\widetilde{U}} - U$ and $[v_2, x_2] = 0$ because $[U, \widehat{\widetilde{U}}] = 0$ in the case $n = 3$. Thus we can take $y_2 = x_2$.

Let us consider $x_3 = \widehat{U}$. We have

$$[x_3, v_3] = [x_3, x_3^2 - (x_1 + x_2)x_3 + x_2x_1] = 0,$$

because the elements $x_1 + x_2 = U + \widehat{\widetilde{U}}$ and $x_2x_1 = \widehat{\widetilde{U}}U$ belong to the center $\mathrm{Cen}(\mathcal{G}_{p,q}) = \mathcal{G}^0_{p,q} \oplus \mathcal{G}^3_{p,q}$, and can take $y_3 = x_3$.

Let us consider $x_4 = \widetilde{U}$. We have

$$[x_4, v_4] = [x_4, x_4^3 - (x_3 + x_2 + x_1)x_4^2 + (x_3x_2 + x_3x_1 + x_2x_1)x_4 - (x_3x_2x_1)] = 0,$$

because the elements $x_1 + x_2 = U + \widehat{\widetilde{U}}$ and $x_2x_1 = \widehat{\widetilde{U}}U$ belong to the center $\mathrm{Cen}(\mathcal{G}_{p,q})$, and $x_3x_4 = x_4x_3$, i.e. $\widehat{U}\widetilde{U} = \widetilde{U}\widehat{U}$ in the case $n = 3$. We take $y_4 = x_4$.

We obtain $y_k = x_k$, $k = 1, 2, 3, 4$ and the following formulas, which are another version of the formulas (11):

$$C_{(1)} = \widetilde{U} + \widehat{U} + \widehat{\widetilde{U}} + U, \qquad C_{(2)} = -(\widetilde{U}\widehat{U} + \widetilde{U}\widehat{\widetilde{U}} + \widetilde{U}U + \widehat{U}\widehat{\widetilde{U}} + \widehat{U}U + \widehat{\widetilde{U}}U),$$
$$C_{(3)} = \widetilde{U}\widehat{U}\widehat{\widetilde{U}} + \widetilde{U}\widehat{U}U + \widetilde{U}\widehat{\widetilde{U}}U + \widehat{U}\widehat{\widetilde{U}}U, \qquad C_{(4)} = -\widetilde{U}\widehat{U}\widehat{\widetilde{U}}U. \tag{21}$$

Note that we obtain these formulas for the element U with invertible expressions v_2, v_3, and v_4 (for other elements U, other sequences x_1, x_2, x_3, x_4 can be considered). Also note that not every sequence y_1, y_2, y_3, y_4 from $\{\widetilde{U}, \widehat{U}, \widehat{\widetilde{U}}, U\}$ gives the correct Vieta's formulas (see Theorem 3 and Lemma 7 in [17], the formulas (11) and (21) are two of several correct forms).

The cases $n \geq 4$

The generalized Vieta's formulas in the cases $n \geq 4$ are more complicated. We use the additional (triangle) operation

$$U^{\triangle} := \sum_{k=0}^{n} (-1)^{\frac{k(k-1)(k-2)(k-3)}{24}} \langle U \rangle_k = \sum_{k=0,1,2,3 \mod 8} \langle U \rangle_k - \sum_{k=4,5,6,7 \mod 8} \langle U \rangle_k. \tag{22}$$

Note that

$$\mathrm{Det}(U^{\triangle}) \neq \mathrm{Det}(U), \qquad \varphi_{U^{\triangle}}(\lambda) \neq \varphi_U(\lambda), \qquad \varphi_U(U^{\triangle}) \neq 0 \tag{23}$$

in the general case (compare with the statements of Theorem 2).

In the case $n = 4$, the generalized Vieta's formulas have the following form

$$\begin{aligned} C_{(1)} &= U + \widehat{\widetilde{U}} + \widehat{U}^{\triangle} + \widetilde{U}^{\triangle}, \qquad C_{(2)} = -(U\widehat{\widetilde{U}} + U\widehat{U}^{\triangle} + U\widetilde{U}^{\triangle} + \widehat{\widetilde{U}}\widehat{U}^{\triangle} + \widehat{\widetilde{U}}\widetilde{U}^{\triangle} + (\widehat{U}\widetilde{U})^{\triangle}), \\ C_{(3)} &= U\widehat{\widetilde{U}}\widehat{U}^{\triangle} + U\widehat{\widetilde{U}}\widetilde{U}^{\triangle} + U(\widehat{U}\widetilde{U})^{\triangle} + \widehat{\widetilde{U}}(\widehat{U}\widetilde{U})^{\triangle}, \qquad C_{(4)} = -U\widehat{\widetilde{U}}(\widehat{U}\widetilde{U})^{\triangle}, \end{aligned} \tag{24}$$

where the coefficients $C_{(k)}$, $k = 1, 2, 3, 4$ are not elementary symmetrical polynomials because of the additional operation of conjugation $\triangle$. These formulas look like the ordinary Vieta's formulas

$$\begin{aligned} C_{(1)} &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \qquad C_{(2)} = -(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4), \\ C_{(3)} &= \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4, \qquad C_{(4)} = -\lambda_1\lambda_2\lambda_3\lambda_4, \end{aligned} \tag{25}$$

if we ignore the operation $\triangle$. The analogues of the formulas (24) for the cases $n = 5, 6$ are presented in [1] (see Theorem 5.1 and Section 8). These formulas also have the form of elementary symmetric polynomials, only if we ignore the operation $\triangle$, and can be interpreted as generalized noncommutative Vieta's formulas. These formulas do not follow directly from the Gelfand – Retakh noncommutative Vieta theorem, it is not easy task to guess the "right" (generic) ordered set of solutions $x_1, x_2, x_3, \ldots x_N$ of the characteristic equation to obtain the elements $y_1, y_2, y_3, \ldots, y_N$ we need in the generalized Vieta's formulas. This is a task for further research.

Conclusions

- We discussed a generalization of Vieta's formulas to the case of geometric algebras of small dimensions:

- We applied the Gelfand - Retakh theorem to the characteristic polynomial in geometric algebras.

- We showed how to express characteristic coefficients in terms of combinations of various involutions of elements.

- We compared the generalized Vieta's formulas with the ordinary Vieta's formulas for eigenvalues. The role of the roots (which are complex scalars) of the characteristic equation is played by some combinations of involutions of elements (which are not scalars).

- We hope that the new approach (related to noncommutative symmetric functions) will help to find more optimized formulas for the determinant and inverse in geometric algebras in the cases $n>6$.

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