Hyperbolic SVD for obtaining solutions of SU(2) Yang-Mills equations

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Let us consider pseudo-Euclidean space $\mathbb{R}^{p,q}$, $n = p + q$ or Euclidean space $\mathbb{R}^n$ of arbitrary finite dimension n. We denote Cartesian coordinates by x^μ, $\mu = 1, \ldots, n$ and partial derivatives by $\partial_\mu = \partial/\partial x^\mu$.
Let us consider

$$\mathrm{G} = \mathrm{SU}(2) = \{S \in \mathrm{Mat}(2, \mathbb{C}) \,|\, S^\dagger S = \mathbf{1}, \det S = 1\},$$
$$\mathfrak{g} = \mathfrak{su}(2) = \{S \in \mathrm{Mat}(2, \mathbb{C}) \,|\, S^\dagger = -S, \mathrm{tr} S = 0\}, \qquad \dim \mathfrak{g} = 3.$$

Let us consider the Yang-Mills equations

$$\partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] = F_{\mu\nu}, \tag{1}$$
$$\partial_\mu F^{\mu\nu} - [A_\mu, F^{\mu\nu}] = J^\nu, \tag{2}$$

where $A_\mu : \mathbb{R}^{p,q} \to \mathfrak{g}$ is the potential, $J^\nu : \mathbb{R}^{p,q} \to \mathfrak{g}$ is the non-Abelian current, $F_{\mu\nu} = -F_{\nu\mu} : \mathbb{R}^{p,q} \to \mathfrak{g}$ is the strength of the Yang-Mills field.
The metric tensor of $\mathbb{R}^{p,q}$ is given by the diagonal matrix

$$\eta = \|\eta_{\mu\nu}\| = \|\eta^{\mu\nu}\| = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$$

with p ones and q minus ones on the diagonal. We can raise or lower indices of components of tensor fields using metric tensor, for example, $F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}$.
We may verify that the current (2) satisfies the non-Abelian conservation law

$$\partial_\nu J^\nu - [A_\nu, J^\nu] = 0. \tag{3}$$

The Yang-Mills equations are gauge invariant w.r.t. the transformations

$$A_\mu \to S^{-1}A_\mu S - S^{-1}\partial_\mu S, \qquad F_{\mu\nu} \to S^{-1}F_{\mu\nu}S, \qquad J^\nu \to S^{-1}J^\nu S, \quad (4)$$

where $\qquad S = S(x) : \mathbb{R}^{p,q} \to \mathrm{G}.$

Particular classes of solutions (monopoles, instantons, merons, etc.):
Wu T.T., Yang C.N. (1968), 't Hooft G. (1974), Polyakov A.M. (1975), Belavin A.A., Polyakov A.M., Schwartz A.S., Tyupkin Yu.S. (1975), Witten E. (1977), Atiyah M., Drinfeld V., Hitchin N., Manin Yu. (1978), de Alfaro V., Fubini S., Furlan G. (1976), . . .

The well-known classes of solutions of the Yang-Mills equations are described in detail in various reviews:

- Actor A., Classical solutions of SU(2) Yang-Mills theories, Rev.Mod.Phys. **51**(1979).
- Zhdanov R.Z., Lahno V.I., Symmetry and Exact Solutions of the Maxwell and SU(2) Yang-Mills Equations, Adv.Chem.Phys. Modern Nonlinear Optics **119** II (2001).

Suppose that A^μ and J^μ do not depend on $x \in \mathbb{R}^{p,q}$. We obtain the following algebraic system of equations

$$[A_\mu, [A^\mu, A^\nu]] = J^\nu, \qquad \nu = 1, \ldots, n. \tag{5}$$

We have the following expression for the strength of the Yang-Mills field

$$F^{\mu\nu} = -[A^\mu, A^\nu]. \tag{6}$$

We want to obtain all solutions $A^\mu \in \mathfrak{su}(2)$ of (5) for arbitrary $J^\nu \in \mathfrak{su}(2)$. Constant solutions of the Yang-Mills equations with zero current $J^\nu = 0$ were considered in the following papers:

Schimming R.: On constant solutions of the Yang-Mills equations. Arch. Math. **24**:2, 65–73 (1988).

Schimming R., Mundt E.: Constant potential solutions of the Yang-Mills equation. J. Math. Phys. **33**, 4250 (1992).

Note that two dimensional Yang-Mills theory is discussed in

Gorsky A., Nekrasov N.: *Hamiltonian systems of Calogero type and two dimensional Yang-Mills theory*, Nucl.Phys. B **414**, 213–238 (1994).

and other papers. We consider the case $n \geq 2$ further for the sake of completeness.

Let us consider the Pauli matrices σ^a, $a = 1, 2, 3$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7}$$

We can take the following basis of the Lie algebra $\mathfrak{su}(2)$

$$\tau^1 = \frac{\sigma^1}{2i}, \qquad \tau^2 = \frac{\sigma^2}{2i}, \qquad \tau^3 = \frac{\sigma^3}{2i}. \tag{8}$$

$$\text{with} \qquad (\tau^a)^\dagger = -\tau^a, \qquad \operatorname{tr} \tau^a = 0, \qquad [\tau^a, \tau^b] = \epsilon^{ab}{}_c \tau^c,$$

where the structural constants of the Lie algebra $\mathfrak{su}(2)$ are the antisymmetric Levi-Civita symbol, $\epsilon^{123} = 1$. For the potential and the current, we have

$$A^\mu = A^\mu{}_a \tau^a, \qquad J^\mu = J^\mu{}_a \tau^a, \qquad A^\mu{}_a, J^\mu{}_a \in \mathbb{R}. \tag{9}$$

Latin indices take values $a = 1, 2, 3$ and Greek indices take values $\mu = 1, 2, \ldots, n$. Substituting (9) into (5), we get

$$\boxed{A_{\mu c} A^\mu{}_a A^\nu{}_b \epsilon^{ab}{}_d \epsilon^{cd}{}_k = J^\nu{}_k, \qquad \nu = 1, \ldots, n, \qquad k = 1, 2, 3.} \tag{10}$$

We obtain $3n$ equations ($k = 1, 2, 3$, $\nu = 1, 2, \ldots, n$) for $3n$ unknown $A^\nu{}_k$ and $3n$ known $J^\nu{}_k$. We can consider (10) as a system of equations for elements of two matrices $A_{n\times 3} = ||A^\nu{}_k||$ and $J_{n\times 3} = ||J^\nu{}_k||$.

Lemma

The system of equations $\boxed{A_{\mu c}A^{\mu}{}_{a}A^{\nu}{}_{b}\epsilon^{ab}{}_{d}\epsilon^{cd}{}_{k} = J^{\nu}{}_{k}, \quad \nu = 1,\ldots,n, \quad k = 1,2,3,}$ *is invariant under the following transformations*

$$
\begin{aligned}
1)\quad & A^{\mu}{}_{b} \to A^{\mu}{}_{a}p^{a}_{b}, \qquad J^{\mu}{}_{b} \to J^{\mu}{}_{a}p^{a}_{b},\\
& \textit{i.e.}\quad A \to AP, \qquad J \to JP, \qquad P = ||p^{a}_{b}|| \in \mathrm{SO}(3),\\
& \textit{where}\quad A_{\mu} \to S^{-1}A_{\mu}S, \qquad J^{\nu} \to S^{-1}J^{\nu}S,\\
& S^{-1}\tau^{a}S = p^{a}_{b}\tau^{b}, \qquad \pm S \in \mathrm{SU}(2) \simeq \mathrm{Spin}(3),\\
2)\quad & A^{\nu}{}_{a} \to q^{\nu}_{\mu}A^{\mu}{}_{a}, \qquad J^{\nu}{}_{a} \to q^{\nu}_{\mu}J^{\mu}{}_{a},\\
& \textit{i.e.}\quad A \to QA, \qquad J \to QJ, \qquad Q = ||q^{\mu}_{\nu}|| \in \mathrm{O}(p,q),\\
& \textit{where}\quad x^{\mu} \to q^{\mu}_{\nu}x^{\nu}.
\end{aligned}
$$

Combining gauge and orthogonal transformations, we conclude that the system is invariant under the transformation

$$
\begin{aligned}
& A^{\nu}_{b} \to q^{\nu}_{\mu}A^{\mu}{}_{a}p^{a}_{b}, \qquad J^{\nu}_{b} \to q^{\nu}_{\mu}J^{\mu}{}_{a}p^{a}_{b},\\
& \text{i.e.}\quad A \to QAP, \qquad J \to QJP, \qquad P \in \mathrm{SO}(3), \quad Q \in \mathrm{O}(p,q).
\end{aligned}
$$

Theorem (Singular Value Decomposition (SVD))

For an arbitrary real matrix $A_{n\times N}$ of the size $n\times N$, there exist orthogonal matrices $L_{n\times n}\in\mathrm{O}(n)$ and $R_{N\times N}\in\mathrm{O}(N)$ such that

$$L^{\mathrm{T}}_{n\times n}A_{n\times N}R_{N\times N}=D_{n\times N}, \tag{11}$$

where

$$D_{n\times N}=\mathrm{diag}(\mu_1,\ldots,\mu_s),\qquad s=\min(n,N),\qquad \mu_1\geq\mu_2\geq\cdots\geq\mu_s\geq 0.$$

The numbers $\mu_1,\ldots,\mu_s$ are called the singular values, the columns l_i of the matrix L are called the left singular vectors, the columns r_i of the matrix R are called the right singular vectors.
The columns of the matrix L are eigenvectors of the matrix AA^{T}, and the columns of the matrix R are eigenvectors of the matrix $A^{\mathrm{T}}A$.
The squares of singular values are eigenvalues of the corresponding matrices.
From this fact, it follows that singular values are uniquely determined.

Theorem (for Euclidean case)

Let $A = ||A^{\nu}{}_{k}||$, $J = ||J^{\nu}{}_{k}||$ *satisfy the system of* $3n$ *cubic equations*

$$\boxed{A_{\mu c} A^{\mu}{}_{a} A^{\nu}{}_{b} \epsilon^{ab}{}_{d} \epsilon^{cd}{}_{k} = J^{\nu}{}_{k}, \qquad \nu = 1, \ldots, n, \qquad k = 1, 2, 3.} \tag{12}$$

Then there exist matrices $P \in \mathrm{SO}(3)$ *and* $Q \in \mathrm{O}(n)$ *such that* QAP *is diagonal. For all such matrices* P *and* Q*, the matrix* QJP *is diagonal too and the system (12) takes the following form under the transformation* $A \to QAP, J \to QJP$*:*

in the case $n = 2$*:*

$$\begin{aligned} -a_1(a_2)^2 &= j_1, \\ -a_2(a_1)^2 &= j_2, \end{aligned} \tag{13}$$

in the cases $n \geq 3$ *:*

$$\begin{aligned} -a_1((a_2)^2 + (a_3)^2) &= j_1, \\ -a_2((a_1)^2 + (a_3)^2) &= j_2, \\ -a_3((a_1)^2 + (a_2)^2) &= j_3. \end{aligned} \tag{14}$$

We denote diagonal elements of the matrix QAP *by* a_1, a_2, a_3 *(or* a_1, a_2*) and diagonal elements of the matrix* QJP *by* j_1, j_2, j_3 *(or* j_1, j_2*).*

Suppose we have known matrix J and want to obtain all solutions A of the system (12). We can always calculate singular values j_1, j_2, j_3 of J and solve the system (14). Finally, we obtain all solutions $A_D = \mathrm{diag}(a_1, a_2, a_3)$ of the system (12) but in some other system of coordinates depending on $Q \in \mathrm{O}(n)$ and with gauge fixing depending on $P \in \mathrm{SO}(3)$. The matrix $A = Q^{-1}A_D P^{-1}$ will be solution of the system (12) in the original system of coordinates and with the original gauge fixing.

Note that $Q^{-1}Q_1^{-1}A_D P_1^{-1}P^{-1}$, for all $Q_1 \in \mathrm{O}(n)$ and $P_1 \in \mathrm{SO}(3)$ such that $Q_1 J_D P_1 = J_D$, $J_D = \mathrm{diag}(j_1, j_2, j_3)$, will be also solutions of the system (12) in the original system of coordinates and with the original gauge fixing because of Lemma.

Example. If the matrix $J = 0$, then all singular values of this matrix equal zero and we can take $Q = P = \mathbf{1}$ for its SVD. We solve the system (14) for $j_1 = j_2 = j_3 = 0$ and obtain all solutions $A_D = \mathrm{diag}(a_1, a_2, a_3)$ of this system. We have $Q_1 J_D P_1 = J_D$ for $J_D = 0$ and any $Q_1 \in \mathrm{O}(n)$, $P_1 \in \mathrm{SO}(3)$. Therefore, the matrices $Q_1 A_D P_1$ for all $Q_1 \in \mathrm{O}(n)$ and $P_1 \in \mathrm{SO}(3)$ will be solutions of the system (12) because of Lemma.

The systems (13), (14) can be rewritten in the following way using $b_k := -a_k$:

$$n=2: \quad b_1 b_2^2 = j_1, \quad b_2 b_1^2 = j_2, \tag{15}$$

$$n \geq 3: \quad b_1(b_2^2+b_3^2) = j_1, \quad b_2(b_1^2+b_3^2) = j_2, \quad b_3(b_1^2+b_2^2) = j_3. \tag{16}$$

The system (16) has the following symmetry (similarly for (15)): if we change the sign of some j_k, $k=1,2,3$, then we must change the sign of the corresponding b_k, $k=1,2,3$. Using SVD, we can always get nonnegative j_k, $k=1,2,3$.

Lemma. The system of equations (15) has the following general solution:

1. in the case $j_1=j_2=0$, has solutions $(b_1,0)$, $(0,b_2)$ for all $b_1, b_2 \in \mathbb{R}$;
2. in the cases $j_1 = 0$, $j_2 \neq 0$; $j_1 \neq 0$, $j_2 = 0$, has no solutions;
3. in the case $j_1 \neq 0$, $j_2 \neq 0$, has a unique solution

$$b_1 = \sqrt[3]{\frac{j_2^2}{j_1}}, \qquad b_2 = \sqrt[3]{\frac{j_1^2}{j_2}}.$$

Lemma. If the system (16) has a solution (b_1,b_2,b_3), where $b_1 \neq 0$, $b_2 \neq 0$, $b_3 \neq 0$, then this system has also a solution $(\frac{K}{b_1}, \frac{K}{b_2}, \frac{K}{b_3})$, where $K=(b_1 b_2 b_3)^{\frac{2}{3}}$.

Example. Let us take $j_1 = 13$, $j_2 = 20$, $j_3 = 15$. Then the system (16) has solutions $(b_1,b_2,b_3) = (1,2,3)$ and $(6^{\frac{2}{3}}, \frac{6^{\frac{2}{3}}}{2}, \frac{6^{\frac{2}{3}}}{3})$.

Lemma. The system of equations

$$b_1(b_2^2+b_3^2)=j_1,\quad b_2(b_1^2+b_3^2)=j_2,\quad b_3(b_1^2+b_2^2)=j_3$$

has the following general solution:
1) in the case $j_1=j_2=j_3=0$, has solutions

$$(b_1,0,0),\quad (0,b_2,0),\quad \text{and}\quad (0,0,b_3),\qquad b_1,b_2,b_3\in\mathbb{R};$$

2) in the cases $j_1=j_2=0$, $j_3\neq 0$ (or similar cases with circular permutation), has no solutions;
3) in the case $j_1\neq 0$, $j_2\neq 0$, $j_3=0$ (or similar cases with circular permutation), has a unique solution

$$b_1=\sqrt[3]{\frac{j_2^2}{j_1}},\qquad b_2=\sqrt[3]{\frac{j_1^2}{j_2}},\qquad b_3=0;$$

4) in the case $j_1=j_2=j_3\neq 0$, has a unique solution

$$b_1=b_2=b_3=\sqrt[3]{\frac{j_1}{2}};$$

5) in the case of not all the same $j_1, j_2, j_3 > 0$, has two solutions

$$(b_{1+}, b_{2+}, b_{3+}), \qquad (b_{1-}, b_{2-}, b_{3-})$$

with the following expression for K from the previous lemma

$$K := b_{1+}b_{1-} = b_{2+}b_{2-} = b_{3+}b_{3-} = (b_{1+}b_{2+}b_{3+})^{\frac{2}{3}} = (b_{1-}b_{2-}b_{3-})^{\frac{2}{3}} :$$

5a) in the case $j_1 = j_2 > j_3 > 0$ (or similar cases with circular permutation)

$$b_{1\pm} = b_{2\pm} = \sqrt[3]{\frac{j_3}{2z_\pm}}, \quad b_{3\pm} = z_\pm b_{1\pm}, \quad z_\pm = \frac{j_1 \pm \sqrt{j_1^2 - j_3^2}}{j_3}.$$

$$\text{Moreover,} \quad z_+ z_- = 1, \quad K = (\frac{j_3}{2})^{\frac{2}{3}}.$$

5b) in the case $j_3 > j_1 = j_2 > 0$ (or similar cases with circular permutation):

$$b_{1\pm} = \frac{1}{w_\pm} b_3, \quad b_{2\pm} = w_\pm b_3, \quad b_{3\pm} = b_3 = \sqrt[3]{\frac{j_1}{s}},$$

$$w_\pm = \frac{s \pm \sqrt{s^2 - 4}}{2}, \quad s = \frac{j_3 + \sqrt{j_3^2 + 8j_1^2}}{2j_1}.$$

$$\text{Moreover,} \quad w_+ w_- = 1, \quad b_{1\pm} = b_{2\mp}, \quad K = (\frac{j_1}{s})^{\frac{2}{3}}.$$

5c) in the case of all different $j_1, j_2, j_3 > 0$:

$$b_{1\pm} = \sqrt[3]{\frac{j_3}{t_0 y_\pm z_\pm}}, \quad b_{2\pm} = y_\pm b_{1\pm}, \quad b_{3\pm} = z_\pm b_{1\pm},$$

$$z_\pm = \sqrt{\frac{y_\pm(j_1 - j_2 y_\pm)}{j_2 - j_1 y_\pm}}, \quad y_\pm = \frac{t_0 \pm \sqrt{t_0^2 - 4}}{2},$$

where $t_0 > 2$ is the solution (it always exists, moreover, it is bigger than $\frac{j_2}{j_1} + \frac{j_1}{j_2}$) of the cubic equation $\quad j_1 j_2 t^3 - (j_1^2 + j_2^2 + j_3^2)t^2 + 4j_3^2 = 0.$

$$\text{Moreover,} \quad y_+ y_- = 1, \quad z_+ z_- = 1, \quad K = (\frac{j_3}{t_0})^{\frac{2}{3}}.$$

Consequences for the strength of the Yang-Mills field.

In the case of the constant potential A^μ of the Yang-Mills field, we have the following expression for the strength

$$F^{\mu\nu} = -[A^\mu, A^\nu] = -[A^\mu{}_a\tau^a, A^\nu{}_b\tau^b] = -A^\mu{}_a A^\nu{}_b \epsilon^{ab}{}_c\tau^c = F^{\mu\nu}{}_c\tau^c. \qquad (17)$$

We take a system of coordinates depending on $Q \in \mathrm{O}(n)$ and a gauge fixing depending on $P \in \mathrm{SO}(3)$ such that the matrices A and J are diagonal.

In the case of dimension $n = 2$:

1. In the case $J = 0$, we have $A = 0$ or $A = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $a \in \mathbb{R} \setminus \{0\}$, $F = 0$.
2. In the case $\mathrm{rank}(J) = 1$, we have no constant solutions.
3. In the case $\mathrm{rank}(J) = 2$, we have a unique solution

$$A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \end{pmatrix}, \qquad a_1 = -\sqrt[3]{\frac{j_2^2}{j_1}}, \qquad a_2 = \sqrt[3]{\frac{j_1^2}{j_2}}.$$

For the strength, we have the following nonzero components of the strength

$$F^{12} = -F^{21} = -\sqrt[3]{j_1 j_2}\tau^3. \qquad (18)$$

We have the following expression for the invariant:

$$F^2 = F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}\sqrt[3]{(j_1 j_2)^2}\mathbf{1}. \qquad (19)$$

In the cases of dimension $n \geq 3$:

1. In the case $J = 0$, we have nonzero potential A^μ but zero strength $F^{\mu\nu} = 0$.
2. In the case $\mathrm{rank}(J) = 1$, we have no constant solutions.
3. In the case $\mathrm{rank}(J) = 2$, we have a unique solution. We have again (18) and (19), where j_1, j_2, and $j_3 = 0$ are singular values of the matrix J.
4. In the case $\mathrm{rank}(J) = 3$, we have one or two solutions.
   1) In the case of all the same singular values $j := j_1 = j_2 = j_3 \neq 0$, we have a unique solution

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \\ \dots & \dots & \dots \\ 0 & 0 & 0 \end{pmatrix}, \qquad a = -\sqrt[3]{\frac{j}{2}}. \tag{20}$$

$$F^{12} = -F^{21} = -\sqrt[3]{\frac{j^2}{4}}\tau^3,\ F^{23} = -F^{32} = -\sqrt[3]{\frac{j^2}{4}}\tau^1,\ F^{31} = -F^{13} = -\sqrt[3]{\frac{j^2}{4}}\tau^2.$$

In this case, we have $F^2 = F_{\mu\nu}F^{\mu\nu} = \frac{-3}{2}\sqrt[3]{\frac{j^4}{16}}\mathbf{1} \neq 0$.

2) In the case of not all the same singular values j_1, j_2, j_3 of the matrix J, we have two different solutions

$$A = \begin{pmatrix} -b_{1\pm} & 0 & 0 \\ 0 & -b_{2\pm} & 0 \\ 0 & 0 & -b_{3\pm} \\ 0 & 0 & 0 \\ \dots & \dots & \dots \\ 0 & 0 & 0 \end{pmatrix}, \tag{21}$$

where $b_{k\pm}$, $k = 1, 2, 3$ are from Case (v) of Lemma. We have

$$F^{12}_{\pm} = -F^{21}_{\pm} = -b_{1\pm}b_{2\pm}\tau^3, \qquad F^{23}_{\pm} = -F^{32}_{\pm} = -b_{2\pm}b_{3\pm}\tau^1,$$

$$F^{31}_{\pm} = -F^{13}_{\pm} = -b_{3\pm}b_{1\pm}\tau^2,$$

$$F^2_{\pm} = -\frac{1}{2}((b_{1\pm}b_{2\pm})^2 + (b_{2\pm}b_{3\pm})^2 + (b_{3\pm}b_{1\pm})^2)\mathbf{1} \neq 0.$$

Lemma. In the case of not all the same j_1, j_2, j_3, the invariant $F^2_\pm$ takes the form:

1. in the case $j_1 = j_2 > j_3 > 0$ (or similar cases with circular permutation):

$$F^2_\pm = \frac{-K^2(1+2z^2_\pm)}{2z_\pm^{\frac{4}{3}}}\mathbf{1}, \qquad F^2_+ \neq F^2_-, \tag{22}$$

$$\text{where} \quad z_\pm = \frac{j_1 \pm \sqrt{j_1^2 - j_3^2}}{j_3}, \quad K = (\frac{j_3}{2})^{\frac{2}{3}}.$$

2. in the case $j_3 > j_1 = j_2 > 0$ (or similar cases with circular permutation):

$$F^2_\pm = \frac{-K^2(s^2-1)}{2}\mathbf{1}, \qquad F^2_+ = F^2_-, \tag{23}$$

$$\text{where} \quad s = \frac{j_3 + \sqrt{j_3^2 + 8j_1^2}}{2j_1} > 2, \quad K = (\frac{j_1}{s})^{\frac{2}{3}}.$$

3. in the case of all different $j_1, j_2, j_3 > 0$:

$$F^2_\pm = \frac{-K^2(y^2_\pm + z^2_\pm + y^2_\pm z^2_\pm)}{2(y_\pm z_\pm)^{\frac{4}{3}}}\mathbf{1}, \qquad F^2_+ \neq F^2_-, \tag{24}$$

where $K = (\frac{j_3}{t_0})^{\frac{2}{3}}$, and $y_\pm$, $z_\pm$, t_0 are from Case 5c) of previous Lemma.

Theorem (HSVD)

For arbitrary matrix $A_{n\times N}\in \mathrm{Mat}(\mathbb{R})$, there exist matrices $R\in \mathrm{O}(N)$ and $L\in \mathrm{O}(p,q)$ such that

$$L^{\mathrm{T}}AR=\Sigma^A,\qquad \Sigma^A=\left(\begin{array}{cccc} X_x & 0 & 0 & 0\\ 0 & 0 & I_d & 0\\ 0 & 0 & 0 & 0\\ \hline 0 & Y_y & 0 & 0\\ 0 & 0 & I_d & 0\\ 0 & 0 & 0 & 0 \end{array}\right)\in \mathrm{Mat}_{n\times N}(\mathbb{R}), \tag{25}$$

where the first block of the matrix Σ^A has p rows and the second block has q rows, X_x and Y_y are diagonal matrices of the corresponding sizes x and y with all positive uniquely determined diagonal elements, I_d is the identity matrix of size d. Here we have

$$d=\mathrm{rank}(A)-\mathrm{rank}(A^{\mathrm{T}}\eta A),\qquad x+y=\mathrm{rank}(A^{\mathrm{T}}\eta A),$$

x is the number of positive eigenvalues of the matrix $A^{\mathrm{T}}\eta A$, y is the number of negative eigenvalues of the matrix $A^{\mathrm{T}}\eta A$.

Theorem. Let $A = ||A^{\nu}{}_{k}||$, $J = ||J^{\nu}{}_{k}||$ satisfy the system of $3n$ cubic equations

$$A_{\mu c} A^{\mu}{}_{a} A^{\nu}{}_{b} \epsilon^{ab}{}_{d} \epsilon^{cd}{}_{k} = J^{\nu}{}_{k}, \qquad \nu = 1, \ldots, n, \qquad k = 1, 2, 3. \tag{26}$$

Then: 1) There exist matrices $P \in \mathrm{SO}(3)$ and $Q \in \mathrm{O}(p, q)$ such that the matrix QAP is in the canonical form (with parameters x_A, y_A, d_A)

$$\Sigma^A = QAP = \left(\begin{array}{cccc} X_{x_A} & 0 & 0 & 0 \\ 0 & 0 & I_{d_A} & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & Y_{y_A} & 0 & 0 \\ 0 & 0 & I_{d_A} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

For all such matrices P and Q, the matrix QJP has the following form

$$QJP = \left(\begin{array}{cccc} Z_{x_A} & 0 & 0 & 0 \\ 0 & 0 & \alpha I_{d_A} & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & W_{y_A} & 0 & 0 \\ 0 & 0 & \alpha I_{d_A} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right),$$

where elements of the diagonal matrices Z and W are real numbers (can be zero), $\alpha \in \mathbb{R}$ (can be zero).

2) For parameters of the matrices A and J, we have:

$$x_J \leq x_A, \qquad y_J \leq y_A, \qquad d_J = d_A > 0 \qquad \text{or} \qquad d_J = 0,\ d_A \geq 0.$$

3) There exist matrices $P \in \mathrm{SO}(3)$ and $Q \in \mathrm{O}(p, q)$ such that the matrix QJP is in the canonical form (with parameters x_J, y_J, d_J)

$$\Sigma^J = QJP = \left(\begin{array}{cccc} X_{x_J} & 0 & 0 & 0 \\ 0 & 0 & I_{d_J} & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & Y_{y_J} & 0 & 0 \\ 0 & 0 & I_{d_J} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

and QAP has the following form

$$QAP = \left(\begin{array}{ccccccc} K_{x_J} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta I_{d_J} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_{x_A - x_J} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{d_A - d_J} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & M_{y_J} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta I_{d_J} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_{y_A - y_J} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{d_A - d_J} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

where $\beta \in \mathbb{R} \setminus \{0\}$; elements of the diagonal matrices K, L, M, N are arbitrary nonzero real numbers.

Summary for the case $\mathbb{R}^{1,1}$:

d_J	x_J	y_J	d_A	x_A	y_A	A	F	F^2
0	1	1	0	1	1	1: see (27)	1: see (28)	1: see (29)
0	1	0				$\varnothing$	$\varnothing$	$\varnothing$
0	0	1				$\varnothing$	$\varnothing$	$\varnothing$
0	0	0	0	0	0	$A=0$	$F=0$	$F^2=0$
			0	1	0	∞: see (30)	$F=0$	$F^2=0$
			0	0	1	∞: see (31)	$F=0$	$F^2=0$
			1	0	0	1: see (32)	$F=0$	$F^2=0$
1	0	0				$\varnothing$	$\varnothing$	$\varnothing$

$$A=\left(\begin{array}{ccc} a_1 & 0 & 0 \\ \hline 0 & a_2 & 0 \end{array}\right), \qquad a_1=\sqrt[3]{\frac{j_2^2}{j_1}}, \qquad a_2=-\sqrt[3]{\frac{j_1^2}{j_2}}. \tag{27}$$

$$F^{12}=-F^{21}=\sqrt[3]{j_1 j_2}\tau^3, \tag{28}$$

$$F^2=F_{\mu\nu}F^{\mu\nu}=\frac{1}{2}\sqrt[3]{(j_1 j_2)^2} I_2 \neq 0. \tag{29}$$

$$A=\left(\begin{array}{ccc} a_1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \qquad a_1\in\mathbb{R}\setminus\{0\}. \tag{30}$$

$$A=\left(\begin{array}{ccc} 0 & 0 & 0 \\ \hline a_1 & 0 & 0 \end{array}\right), \qquad a_1\in\mathbb{R}\setminus\{0\}. \tag{31}$$

$$A=\left(\begin{array}{ccc} 1 & 0 & 0 \\ \hline 1 & 0 & 0 \end{array}\right). \tag{32}$$

Summary for the case $\mathbb{R}^{p,q}$, $p+q\geq 2$:

p,q	d_J	x_J	y_J	add.cond.	d_A	x_A	y_A	A	F	F^2
$p\geq 3, q\geq 1$	0	3	0	$j_1=j_2=j_3$	0	3	0	1:...	1:...	1:...
$p\geq 3, q\geq 1$	0	3	0	$j_1=j_2>j_3$	0	3	0	2:...	2:...	2:...
$p\geq 3, q\geq 1$	0	3	0	$j_3>j_1=j_2$	0	3	0	2:...	2:...	1:...
$p\geq 3, q\geq 1$	0	3	0	all different j_1,j_2,j_3	0	3	0	2:...	2:...	2:...
$p\geq 1, q\geq 3$	0	0	3	$j_1=j_2=j_3$	0	0	3	1:...	1:...	1:...
$p\geq 1, q\geq 3$	0	0	3	$j_1=j_2>j_3$	0	0	3	2:...	2:...	2:...
$p\geq 1, q\geq 3$	0	0	3	$j_3>j_1=j_2$	0	0	3	2:...	2:...	1:...
$p\geq 1, q\geq 3$	0	0	3	all different j_1,j_2,j_3	0	0	3	2:...	2:...	2:...
$p\geq 2, q\geq 1$	0	2	1	$j_1=j_2<\frac{j_3}{2\sqrt{2}}$	0	2	1	6:...	6:...	4 or 3:...
$p\geq 2, q\geq 1$	0	2	1	$j_1=j_2=\frac{j_3}{2\sqrt{2}}$	0	2	1	4:...	4:...	$F^2=0$ and 2:...
$p\geq 2, q\geq 1$	0	2	1	$j_1=j_2>\frac{j_3}{2\sqrt{2}}$	0	2	1	2:...	2:...	2:...
$p\geq 2, q\geq 1$	0	2	1	$j_1\neq j_2$, $j_3^{\frac{2}{3}}>j_2^{\frac{2}{3}}+j_1^{\frac{2}{3}}$	0	2	1	6:...	6:...	3-6:...
$p\geq 2, q\geq 1$	0	2	1	$j_1\neq j_2$, $j_3^{\frac{2}{3}}=j_2^{\frac{2}{3}}+j_1^{\frac{2}{3}}$	0	2	1	4:...	4:...	$F^2=0$ and 3:...
$p\geq 2, q\geq 1$	0	2	1	$j_1\neq j_2$, $j_3^{\frac{2}{3}}<j_2^{\frac{2}{3}}+j_1^{\frac{2}{3}}$	0	2	1	2:...	2:...	2:...
$p\geq 1, q\geq 2$	0	1	2	$j_3=j_1<\frac{j_2}{2\sqrt{2}}$	0	1	2	6:...	6:...	4 or 3:...
$p\geq 1, q\geq 2$	0	1	2	$j_3=j_1=\frac{j_2}{2\sqrt{2}}$	0	1	2	4:...	4:...	$F^2=0$ and 2:...
$p\geq 1, q\geq 2$	0	1	2	$j_3=j_1>\frac{j_2}{2\sqrt{2}}$	0	1	2	2:...	2:...	2:...
$p\geq 1, q\geq 2$	0	1	2	$j_3\neq j_1$, $j_2^{\frac{2}{3}}>j_3^{\frac{2}{3}}+j_1^{\frac{2}{3}}$	0	1	2	6:...	6:...	3-6:...
$p\geq 1, q\geq 2$	0	1	2	$j_3\neq j_1$, $j_2^{\frac{2}{3}}=j_3^{\frac{2}{3}}+j_1^{\frac{2}{3}}$	0	1	2	4:...	4:...	$F^2=0$ and 3:...
$p\geq 1, q\geq 2$	0	1	2	$j_3\neq j_1$, $j_2^{\frac{2}{3}}<j_3^{\frac{2}{3}}+j_1^{\frac{2}{3}}$	0	1	2	2:...	2:...	2:...

p, q	d_J	x_J	y_J	**add.cond.**	d_A	x_A	y_A	A	F	F^2
$p \geq 2, q \geq 1$	0	2	0		0	2	0	1:...	1:...	1:...
$p \geq 1, q \geq 2$	0	0	2		0	0	2	1:...	1:...	1:...
$p \geq 1, q \geq 1$	0	1	1		0	1	1	1:...	1:...	1:...
$p \geq 2, q \geq 2$	0	1	1	$j_1 = j_2$	0	2	2	1:...	1:...	1:...
$p \geq 2, q \geq 1$	0	1	1	$j_2 > j_1$	0	2	1	4:...	4:...	2:...
$p \geq 1, q \geq 2$	0	1	1	$j_1 > j_2$	0	1	2	4:...	4:...	2:...
$p \geq 1, q = 1$	0	1	0					$\varnothing$	$\varnothing$	$\varnothing$
$p \geq 1, q \geq 2$	0	1	0		0	1	2	4:...	4:...	1:...
$p = 1, q \geq 1$	0	0	1					$\varnothing$	$\varnothing$	$\varnothing$
$p \geq 2, q \geq 1$	0	0	1		0	2	1	4:...	4:...	1:...
$p \geq 1, q \geq 1$	0	0	0		0	0	0	$A = 0$	$F = 0$	$F^2 = 0$
$p \geq 1, q \geq 1$	0	0	0		0	1	0	∞:...	$F = 0$	$F^2 = 0$
$p \geq 1, q \geq 1$	0	0	0		0	0	1	∞:...	$F = 0$	$F^2 = 0$
$p \geq 1, q \geq 1$	0	0	0		1	0	0	1:...	$F = 0$	$F^2 = 0$
$p \geq 2, q \geq 2$	0	0	0		2	0	0	1:...	1:...	$F^2 = 0$
$p \geq 3, q \geq 3$	0	0	0		3	0	0	1:...	1:...	$F^2 = 0$
$p \geq 3, q \geq 1$	1	2	0		1	2	0	1:...	1:...	1:...
$p \geq 1, q \geq 3$	1	0	2		1	0	2	1:...	1:...	1:...
$p \geq 2, q \geq 2$	1	1	1	$j_1 = j_2$				$\varnothing$	$\varnothing$	$\varnothing$
$p \geq 2, q \geq 2$	1	1	1	$j_1 \neq j_2$	1	1	1	1:...	1:...	1:...
$p \geq 2, q \geq 1$	1	1	0					$\varnothing$	$\varnothing$	$\varnothing$
$p \geq 1, q \geq 2$	1	0	1					$\varnothing$	$\varnothing$	$\varnothing$
$p = 1, q = 1$	1	0	0					$\varnothing$	$\varnothing$	$\varnothing$
$p \geq 2, q \geq 1$	1	0	0		1	1	0	∞:...	∞:...	$F^2 = 0$
$p \geq 1, q \geq 2$	1	0	0		1	0	1	∞:...	∞:...	$F^2 = 0$
$p \geq 3, q \geq 2$	2	1	0					$\varnothing$	$\varnothing$	$\varnothing$
$p \geq 2, q \geq 3$	2	0	1					$\varnothing$	$\varnothing$	$\varnothing$
$p = 2, q = 2$	2	0	0					$\varnothing$	$\varnothing$	$\varnothing$
$p \geq 3, q \geq 2$	2	0	0		2	1	0	∞:...	∞:...	$F^2 = 0$
$p \geq 2, q \geq 3$	2	0	0		2	0	1	∞:...	∞:...	$F^2 = 0$
$p \geq 3, q \geq 3$	3	0	0					$\varnothing$	$\varnothing$	$\varnothing$

- We obtain all constant solutions of $\mathrm{SU}(2)$ Yang-Mills equations in $\mathbb{R}^n$ for arbitrary current.
- We prove that the number (0, 1, or 2) of solutions in terms of the strength F depends on the singular values of the matrix J. The explicit form of these solutions and the invariant F^2 can always be written using these singular values.
- We have analogous results for the case of pseudo-Euclidean space $\mathbb{R}^{p,q}$ of arbitrary finite dimension $n = p + q$, in particular, for the case of Minkowski space $\mathbb{R}^{1,3}$. We use hyperbolic SVD.
- This will allow us to obtain all constant solutions of Dirac-Yang-Mills equations.
- We can consider nonconstant solutions of the Yang-Mills equations in the form of series of perturbation theory using all constant solutions as a zeroth approximation. The problem reduces to solving systems of linear partial differential equations.
- We hope that the results can be useful for solving some problems in Particle physics, in particular, in describing physical vacuum.

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Thank you for your attention!