On grade automorphism and unitary groups in ternary Clifford Algebras

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Ternary Clifford algebra with two generators

Let us consider the ternary Clifford algebra with two generators $C\ell_2^{\frac{1}{3}}$ [1]. The generators satisfy

$$e_1^3 = e_2^3 = e, \qquad e \equiv 1, \tag{1}$$

$$e_1 e_2 = \omega e_2 e_1, \qquad \omega = e^{\frac{2\pi i}{3}}. \tag{2}$$

An arbitrary element $U \in C\ell_2^{\frac{1}{3}}$ has the form

$$U = \sum_{j,k=0}^{2} u_{jk} e_1^j e_2^k = u_{00} e + u_{10} e_1 + u_{01} e_2 + u_{20} e_1^2 + u_{02} e_2^2 + u_{11} e_1 e_2$$
$$+ u_{21} e_1^2 e_2 + u_{12} e_1 e_2^2 + u_{22} e_1^2 e_2^2, \qquad u_{jk} \in \mathbb{C}. \tag{3}$$

The multiplication table is the following (see Table 1).

1st \ 2nd	e_1	e_2	e_1^2	e_2^2	$e_1 e_2$	$e_1^2 e_2$	$e_1 e_2^2$	$e_1^2 e_2^2$
e_1	e_1^2	$e_1 e_2$	e	$e_1 e_2^2$	$e_1^2 e_2$	e_2	$e_1^2 e_2^2$	e_2^2
e_2	$\omega^2 e_1 e_2$	e_2^2	$\omega e_1^2 e_2$	e	$\omega^2 e_1 e_2^2$	$\omega e_1^2 e_2^2$	$\omega^2 e_1$	$\omega^2 e_1^2$
e_1^2	e	$e_1^2 e_2$	e_1	$e_1^2 e_2^2$	e_2	$e_1 e_2$	e_2^2	$e_1 e_2^2$
e_2^2	$\omega e_1 e_2^2$	e	$\omega^2 e_1^2 e_2^2$	e_2	ωe_1	$\omega^2 e_1^2$	$\omega e_1 e_2$	$\omega e_1^2 e_2$
$e_1 e_2$	$\omega^2 e_1^2 e_2$	$e_1 e_2^2$	ωe_2	e_1	$\omega^2 e_1^2 e_2^2$	ωe_2^2	$\omega^2 e_1^2$	ωe
$e_1^2 e_2$	$\omega^2 e_2$	$e_1^2 e_2^2$	$\omega e_1 e_2$	e_1^2	$\omega^2 e_2^2$	$\omega e_1 e_2^2$	$\omega^2 e$	ωe_1
$e_1 e_2^2$	$\omega e_1^2 e_2^2$	e_1	$\omega^2 e_2^2$	$e_1 e_2$	ωe_1^2	$\omega^2 e$	$\omega e_1^2 e_2$	ωe_2
$e_1^2 e_2^2$	ωe_2^2	e_1^2	$\omega^2 e_1 e_2^2$	$e_1^2 e_2$	ωe	$\omega^2 e_1$	ωe_2	$\omega e_1 e_2$

Table: Multiplication table in $C\ell_2^{\frac{1}{3}}$.

Let us consider the following explicit matrix representation (isomorphism)

$$\beta : C\ell_2^{\frac{1}{3}} \to \mathrm{Mat}(3, \mathbb{C}).$$

We use the following matrices

$$\beta(e) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \beta(e_1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \beta(e_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix},$$

$$\beta(e_1^2) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \beta(e_2^2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{bmatrix}, \beta(e_1 e_2) = \begin{bmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\beta(e_1^2 e_2) = \begin{bmatrix} 0 & 0 & \omega^2 \\ 1 & 0 & 0 \\ 0 & \omega & 0 \end{bmatrix}, \beta(e_1 e_2^2) = \begin{bmatrix} 0 & \omega^2 & 0 \\ 0 & 0 & \omega \\ 1 & 0 & 0 \end{bmatrix}, \beta(e_1^2 e_2^2) = \begin{bmatrix} 0 & 0 & \omega \\ 1 & 0 & 0 \\ 0 & \omega^2 & 0 \end{bmatrix}$$

For an arbitrary $U \in C\ell_2^{\frac{1}{3}}$, we get

$$\beta(U) = \begin{bmatrix} u_{00} + u_{01} + u_{02} & u_{10} + \omega u_{11} + \omega^2 u_{12} & u_{20} + \omega u_{22} + \omega^2 u_{21} \\ u_{20} + u_{21} + u_{22} & u_{00} + \omega u_{01} + \omega^2 u_{02} & u_{10} + \omega u_{12} + \omega^2 u_{11} \\ u_{10} + u_{11} + u_{12} & u_{20} + \omega u_{21} + \omega^2 u_{22} & u_{00} + \omega u_{02} + \omega^2 u_{01} \end{bmatrix}. \quad (4)$$

Ternary Clifford algebra with d generators

Let us consider the ternary Clifford algebra $C\ell^{\frac{1}{3}}_d$ with d generators that satisfy

$$e_j^3 = e, \qquad j = 1, \ldots, d, \tag{5}$$

$$e_i e_j = \omega e_j e_i, \qquad i < j, \qquad \omega = e^{\frac{2\pi i}{3}}. \tag{6}$$

We have $\dim(C\ell^{\frac{1}{3}}_d) = 3^d$. An arbitrary element $U \in C\ell^{\frac{1}{3}}_d$ has the form

$$U = \sum_{j_1, \ldots, j_d = 0}^{2} u_{j_1 \ldots j_d} e_1^{j_1} \cdots e_d^{j_d} \tag{7}$$

$$= u_{0\ldots0} e + u_{10\ldots0} e_1 + \cdots + u_{0\ldots01} e_d + \cdots + u_{2\ldots2} e_1^2 \cdots e_d^2$$

We have the following isomorphisms:

$$C\ell^{\frac{1}{3}}_d \cong \mathrm{Mat}(3^{\frac{d}{2}}, \mathbb{C}), \qquad d = 0 \mod 2; \tag{8}$$

$$C\ell^{\frac{1}{3}}_d \cong \mathrm{Mat}(3^{\frac{d-1}{2}}, \mathbb{C}) \oplus \mathrm{Mat}(3^{\frac{d-1}{2}}, \mathbb{C}) \oplus \mathrm{Mat}(3^{\frac{d-1}{2}}, \mathbb{C}), \quad d = 1 \mod 2. \tag{9}$$

The center of $C\ell^{\frac{1}{3}}_d$ has the form [1]

$$\mathrm{cen}(C\ell^{\frac{1}{3}}_d) = \{\lambda e, \quad \lambda \in \mathbb{C}\} \tag{10}$$

in the case of even d and

$$\mathrm{cen}(C\ell^{\frac{1}{3}}_d) = \{\lambda e + \alpha e^1_1 e^2_2 e^1_3 \cdots e^1_d + \beta e^2_1 e^1_2 e^2_3 \cdots e^2_d, \qquad \lambda, \alpha, \beta \in \mathbb{C}\} \tag{11}$$

in the case of odd d.

Let us introduce the operation [1] in $C\ell^{\frac{1}{3}}_d$:

$$U^\dagger := U|_{e_i^j \to e_i^{-j}}. \tag{12}$$

We have the properties

$$(U^\dagger)^\dagger = U, \qquad (U+V)^\dagger = U^\dagger + V^\dagger, \qquad (\alpha U)^\dagger = \alpha U^\dagger, \qquad (UV)^\dagger = U^\dagger V^\dagger.$$

In the case $d=2$, for an arbitrary U of the form (3), we have

$$U^\dagger = \sum_{j,k=0}^{2} u_{jk}(e_1^{-j} e_2^{-k}) = u_{00}e + u_{10}e_1^2 + u_{01}e_2^2 + u_{20}e_1 + u_{02}e_2 + \tag{13}$$

$$+u_{11}e_1^2e_2^2 + u_{21}e_1e_2^2 + u_{12}e_1^2e_2 + u_{22}e_1e_2,$$

and

$$U^\dagger = T^{-1}UT, \tag{14}$$

where

$$T = \frac{1}{3}(e + e_1 + e_2 + e_1^2 + e_2^2 + \omega e_1 e_2 + \omega^2 e_1^2 e_2 + \omega^2 e_1 e_2^2 + \omega e_1^2 e_2^2), \qquad T^2 = e.$$

We have

$$\beta(T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\beta(U^\dagger) = \begin{bmatrix} u_{00} + u_{01} + u_{02} & u_{20} + \omega u_{22} + \omega^2 u_{21} & u_{10} + \omega u_{11} + \omega^2 u_{12} \\ u_{10} + u_{12} + u_{11} & u_{00} + \omega u_{02} + \omega^2 u_{01} & u_{20} + \omega u_{21} + \omega^2 u_{22} \\ u_{20} + u_{22} + u_{21} & u_{10} + \omega u_{12} + \omega^2 u_{11} & u_{00} + \omega u_{01} + \omega^2 u_{02} \end{bmatrix},$$

and

$$\det(U^\dagger) = \det(U).$$

Let us consider the subspace $C\ell_d^{\frac{1}{3},k}$ of grade k with the basis elements that are products of k generators. We have

$$C\ell_d^{\frac{1}{3}} = \bigoplus_{k=0}^{2d} C\ell_d^{\frac{1}{3},k}. \qquad (15)$$

It is easily verified that

$$C\ell_d^{\frac{1}{3},k} C\ell_d^{\frac{1}{3},j} \subseteq \bigoplus_{s=0}^{[\frac{k+j}{3}]} C\ell_d^{\frac{1}{3},k+j-3s} \qquad (16)$$

We have

$$C\ell_d^{\frac{1}{3}} = C\ell_d^{\frac{1}{3},(0)} \oplus C\ell_d^{\frac{1}{3},(1)} \oplus C\ell_d^{\frac{1}{3},(2)}, \qquad C\ell_d^{\frac{1}{3},(k)} := \bigoplus_{j=k \bmod 3} C\ell_d^{\frac{1}{3},j}. \qquad (17)$$

We have $\mathbb{Z}_3$-grading:

$$C\ell_d^{\frac{1}{3},(k)} C\ell_d^{\frac{1}{3},(j)} \subseteq C\ell_d^{\frac{1}{3},(k+j) \,\mathrm{mod} 3} \qquad (18)$$

Example. Suppose $T \in C\ell_d^{\frac{1}{3}}$ is invertible. If $T \in C\ell_d^{\frac{1}{3},(k)}$, $k = 0, 1, 2$, then $T^{-1} \in C\ell_d^{\frac{1}{3},(-k) \bmod 3}$.

We use the notation

$$U = \sum_{j=0}^{2} U_{(j)}, \qquad U_{(j)} \in C\ell_d^{\frac{1}{3},(k)}. \tag{19}$$

Let us consider the following operation (*grade authomorphism*)

$$\widehat{U} := U_{(0)} + \omega U_{(1)} + \omega^2 U_{(2)}, \qquad U \in C\ell_d^{\frac{1}{3}}. \tag{20}$$

This operation is not an involution because $\widehat{\widehat{U}} \neq U$.
We have

$$C\ell_d^{\frac{1}{3},(k)} = \{U \in C\ell_d^{\frac{1}{3}} : \widehat{U} = \omega^k U\}, \qquad k = 0, 1, 2. \tag{21}$$

Theorem

The operation (20) has the properties

$$\widehat{\widehat{\widehat{U}}} = U, \qquad (\widehat{U + V}) = \widehat{U} + \widehat{V}, \qquad \widehat{(\alpha U)} = \alpha \widehat{U}, \qquad \widehat{UV} = \widehat{U}\widehat{V}. \tag{22}$$

Theorem

In the case of $C\ell_d^{\frac{1}{3}}$ with even d, the grade automorphism is an inner automorphism:

$$\widehat{U}=T^{-1}UT=SUS^{-1},\quad T=e_1^2e_2e_3^2\cdots e_d,\quad S=e_1e_2^2e_3\cdots e_d^2,\quad T^3=S^3=e.$$

In the particular case $d=2$, we have

$$\widehat{U}=u_{00}e+u_{10}\omega e_1+u_{01}\omega e_2+u_{20}\omega^2e_1^2+u_{02}\omega^2e_2^2+u_{11}\omega^2e_1e_2 \tag{23}$$
$$+u_{21}e_1^2e_2+u_{12}e_1e_2^2+u_{22}\omega^2e_1^2e_2^2,\qquad u_{jk}\in\mathbb{C}, \tag{24}$$

$$\beta(\widehat{U})=\begin{bmatrix} u_{00}+\omega u_{01}+\omega^2u_{02} & \omega u_{10}+u_{11}+\omega^2u_{12} & \omega^2u_{20}+\omega^2u_{22}+\omega^2u_{21}\\ \omega^2u_{20}+u_{21}+\omega u_{22} & u_{00}+\omega^2u_{01}+\omega u_{02} & \omega u_{10}+\omega u_{12}+\omega u_{11}\\ \omega u_{10}+\omega^2u_{11}+u_{12} & \omega^2u_{20}+\omega u_{21}+u_{22} & u_{00}+u_{02}+u_{01}\end{bmatrix},$$

and

$$\widehat{U}=T^{-1}UT=SUS^{-1},\qquad T=e_1^2e_2,\qquad S=e_1e_2^2,\qquad T^3=S^3=e. \tag{26}$$

We have

$$\det(U)=\det(\widehat{U}).$$

Let us consider the following operation of Hermitian conjugation [1]:

$$\overline{U} := U|_{u_{jk}\to\overline{u_{jk}},\, e_1^j e_2^k \to (e_1^j e_2^k)^{-1}}, \qquad \forall U \in C\ell_2^{\frac{1}{3}}. \tag{27}$$

For an arbitrary U of the form (3), we have

$$\overline{U} = \sum_{j,k=0}^{2} \overline{u_{jk}} (e_1^j e_2^k)^{-1} = \overline{u_{00}} e + \overline{u_{10}} e_1^2 + \overline{u_{01}} e_2^2 + \overline{u_{20}} e_1 + \overline{u_{02}} e_2 + \tag{28}$$

$$+ \overline{u_{11}} \omega^2 e_1^2 e_2^2 + \overline{u_{21}} \omega e_1 e_2^2 + \overline{u_{12}} \omega e_1^2 e_2 + \overline{u_{22}} \omega^2 e_1 e_2. \tag{29}$$

We have the properties

$$\overline{\overline{U}} = U, \qquad \overline{U+V} = \overline{U} + \overline{V}, \qquad \overline{\alpha U} = \overline{\alpha}\, \overline{U}, \qquad \overline{UV} = \overline{V}\ \overline{U}. \tag{30}$$

For the matrix representation β of minimal dimension (31), we have

$$(\beta(U))^{\mathrm{H}} = \beta(\overline{U}), \qquad U \in C\ell_2^{\frac{1}{3}},$$

where H is the Hermitian transpose. In particular, we have

$$\det(U) = \overline{\det(\overline{U})}, \qquad U \in C\ell_2^{\frac{1}{3}}.$$

From (27), we get the inner product

$$U \cdot V := \langle \overline{U} V \rangle_0 = \sum_A \overline{u_A} v_A$$

and the norm

$$||U|| := \sqrt{\overline{U} \cdot U} = \sqrt{\sum_A |u_A|^2} \geq 0.$$

Lemma

We have $\mathrm{tr}(\beta(U)) = 3\langle U\rangle_0$ *for* $\forall U \in C\ell_2^{\frac{1}{3}}$.

Let us introduce the notion of determinant of multivector $U \in C\ell_2^{\frac{1}{3}}$:

$$\det(U) := \det(\beta(U)) \in \mathbb{C}$$

for an arbitrary matrix representation of minimal dimension

$$\beta : C\ell_2^{\frac{1}{3}} \to \mathrm{Mat}(3, \mathbb{C}). \qquad (31)$$

Lemma

The determinant is well-defined, i.e. it does not depend on the matrix representation β.

For (3), we get

$$\begin{aligned}\det(U) &= u_{00}^3 + u_{10}^3 + u_{01}^3 + u_{20}^3 + u_{02}^3 + u_{11}^3 + u_{21}^3 + u_{12}^3 + u_{22}^3 \qquad (32)\\ &-3(u_{00}u_{01}u_{02} + u_{10}u_{11}u_{12} + u_{00}u_{10}u_{20} + u_{01}u_{11}u_{21} + u_{02}u_{12}u_{22}\\ &+u_{20}u_{21}u_{22}) - 3\omega(u_{01}u_{12}u_{20} + u_{02}u_{10}u_{21} + u_{00}u_{11}u_{22})\\ &-3\omega^2(u_{02}u_{11}u_{20} + u_{00}u_{12}u_{21} + u_{01}u_{10}u_{22}) \in \mathbb{C}.\end{aligned}$$

Let us introduce the operation

$$\underline{U} := \langle U\rangle_0 - \sum_{k=1}^{4}\langle U\rangle_k = 2\langle U\rangle_0 - U, \qquad U \in C\ell_2^{\frac{1}{3}}, \qquad \langle U\rangle_k \in C\ell_2^{\frac{1}{3},k}.$$

We have

$$\langle U\rangle_0 = \frac{U+\underline{U}}{2} = \frac{\mathrm{tr}(\beta(U))}{3}.$$

Theorem

We have the properties:

$$\underline{\underline{U}} = U, \qquad \underline{U+V} = \underline{U}+\underline{V}, \qquad \underline{\alpha U} = \alpha\underline{U}, \qquad \underline{UV}U = U\underline{VU}.$$

Now let us introduce a characteristic polynomial of $U \in C\ell_2^{\frac{1}{3}}$:

$$\varphi_U(\lambda) := \det(\lambda e - U) = \lambda^3 - C_{(1)}\lambda^2 - C_{(2)}\lambda - C_{(3)} \in \mathbb{C}, \qquad \lambda \in \mathbb{C}. \tag{33}$$

Theorem

For the coefficients $C_{(k)}$, $k = 1, 2, 3$ of characteristic polynomial of an arbitrary multivector $U \in C\ell_2^{\frac{1}{3}}$, we have:

$$\mathrm{tr}(U) = C_{(1)} = \frac{3}{2}(U + \underline{U}), \tag{34}$$

$$C_{(2)} = -\frac{3}{8}U^2 - \frac{3}{8}\underline{U^2} - \frac{9}{8}U\underline{U} - \frac{9}{8}\underline{U\underline{U}}, \tag{35}$$

$$\det(U) = C_{(3)} = U(-\frac{1}{8}U^2 - \frac{3}{8}U\underline{U} + \frac{3}{8}\underline{U^2} + \frac{9}{8}\underline{U\underline{U}}). \tag{36}$$

In particular, we have

$$\det(e) = \det(e_1) = \det(e_2) = \det(e_1^2) = \det(e_2^2) = \det(e_1 e_2)$$
$$= \det(e_1^2 e_2) = \det(e_1 e_2^2) = \det(e_1^2 e_2^2) = 1.$$

Corollary

In the case $\det(U) \neq 0$, *we have an explicit formula for the inverse:*

$$U^{-1} = \frac{-U^2 - 3U\underline{U} + 3\underline{U^2} + 9\underline{U\underline{U}}}{8\det(U)}, \qquad U \in C\ell_2^{\frac{\mathbf{1}}{\mathbf{3}}}, \tag{37}$$

where

$$\det(U) = U(-\frac{1}{8}U^2 - \frac{3}{8}U\underline{U} + \frac{3}{8}\underline{U^2} + \frac{9}{8}\underline{U\underline{U}}) \in \mathbb{C}.$$

Let us consider the Lie groups

$$\mathrm{U}C\ell_2^{\frac{1}{3}} := \{U \in C\ell_2^{\frac{1}{3}} : \quad \overline{U}U = e\}, \tag{38}$$

$$\mathrm{SU}C\ell_2^{\frac{1}{3}} := \{U \in C\ell_2^{\frac{1}{3}} : \quad \overline{U}U = e, \quad \det(U) = 1\}, \tag{39}$$

where $\det(U) = U(-\frac{1}{8}U^2 - \frac{3}{8}U\underline{U} + \frac{3}{8}\underline{U^2} + \frac{9}{8}\underline{U\underline{U}})$. We call these two groups *unitary and special unitary groups* in ternary Clifford algebra $C\ell_2^{\frac{1}{3}}$.

Theorem

We have the following isomorphisms

$$\mathrm{U}C\ell_2^{\frac{1}{3}} \simeq \mathrm{U}(3) = \{A \in \mathrm{Mat}(3, \mathbb{C}) : \quad A^{\mathrm{H}}A = I\}, \tag{40}$$

$$\mathrm{SU}C\ell_2^{\frac{1}{3}} \simeq \mathrm{SU}(3) = \{A \in \mathrm{Mat}(3, \mathbb{C}) : \quad A^{\mathrm{H}}A = I, \quad \det(A) = 1\}. \tag{41}$$

We have

$$\dim(\mathrm{U}C\ell_2^{\frac{1}{3}}) = 9, \qquad \dim(\mathrm{SU}C\ell_2^{\frac{1}{3}}) = 8. \tag{42}$$

Corollary

The corresponding Lie algebras

$$\mathfrak{u}C\ell_2^{\frac{1}{3}} := \{U \in C\ell_2^{\frac{1}{3}} : \quad \overline{U} = -U\}, \qquad \dim(\mathfrak{u}C\ell_2^{\frac{1}{3}}) = 8, \tag{43}$$

$$\mathfrak{su}C\ell_2^{\frac{1}{3}} := \{U \in C\ell_2^{\frac{1}{3}} : \quad \overline{U} = -U, \quad \mathrm{tr}(U) = 0\}, \quad \dim(\mathfrak{su}C\ell_2^{\frac{1}{3}}) = 8, \tag{44}$$

are isomorphic to

$$\mathfrak{u}C\ell_2^{\frac{1}{3}} \simeq \mathfrak{u}(3) = \{A \in \mathrm{Mat}(3,\mathbb{C}) : \quad A^{\mathrm{H}} = -A\}, \tag{45}$$

$$\mathfrak{su}C\ell_2^{\frac{1}{3}} \simeq \mathfrak{su}(3) = \{A \in \mathrm{Mat}(3,\mathbb{C}) : \quad A^{\mathrm{H}} = -A, \quad \mathrm{tr}(A) = 0\}. \tag{46}$$

Theorem

The basis of $\mathfrak{su}C\ell_2^{\frac{1}{3}}$ *is*

$$\begin{aligned} &\tau_1 = e_1 - e_1^2, \quad \tau_2 = i(e_1 + e_1^2), \quad \tau_3 = e_2 - e_2^2, \quad \tau_4 = i(e_2 + e_2^2), \\ &\tau_5 = e_1 e_2 - \omega^2 e_1^2 e_2^2, \quad \tau_6 = i(e_1 e_2 + \omega^2 e_1^2 e_2^2), \\ &\tau_7 = e_1^2 e_2 - \omega e_1 e_2^2, \quad \tau_8 = i(e_1^2 e_2 + \omega e_1 e_2^2). \end{aligned} \tag{47}$$

The basis of $\mathfrak{u}C\ell_2^{\frac{1}{3}}$ is

$$\tau_0 = ie, \qquad \tau_j, \quad j = 1, \ldots, 8. \tag{48}$$

Note that all τ_j, $j = 0, 1, \ldots, 8$ are anti-Hermitian $\overline{\tau_j} = -\tau_j$.
We have (we use notation $\beta_j := \beta(\tau_j)$):

$$\beta_0 = \begin{bmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{bmatrix}, \beta_1 = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \beta_2 = \begin{bmatrix} 0 & i & i \\ i & 0 & i \\ i & i & 0 \end{bmatrix},$$

$$\beta_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sqrt{3}i & 0 \\ 0 & 0 & -\sqrt{3}i \end{bmatrix}, \beta_4 = \begin{bmatrix} 2i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{bmatrix}, \beta_5 = \begin{bmatrix} 0 & \omega & -1 \\ -\omega^2 & 0 & \omega^2 \\ 1 & -\omega & 0 \end{bmatrix}$$

$$\beta_6 = \begin{bmatrix} 0 & i\omega & i \\ i\omega^2 & 0 & i\omega^2 \\ i & i\omega & 0 \end{bmatrix}, \beta_7 = \begin{bmatrix} 0 & -1 & \omega^2 \\ 1 & 0 & -\omega^2 \\ -\omega & \omega & 0 \end{bmatrix}, \beta_8 = \begin{bmatrix} 0 & i & i\omega^2 \\ i & 0 & i\omega^2 \\ i\omega & i\omega & 0 \end{bmatrix}$$

All these matrices are anti-Hermitian $\beta_j^{\mathrm{H}} = -\beta_j$, $j = 0, 1, \ldots, 8$.

Another basis of $\mathfrak{su}(3)$

$$\theta_j = i\lambda_j, \qquad j = 1, \ldots, 9$$

is constructed using the well-known Gell–Mann matrices

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$\lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \quad \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$
$$\lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

One can write down an explicit relation between these two bases.

In the case of the ternary Clifford algebras $C\ell^{\frac{1}{3}}_d$, we have the following isomorphisms with unitary groups.
If d is even, then

$$\mathrm{U}C\ell^{\frac{1}{3}}_d = \{U \in C\ell^{\frac{1}{3}}_d : \quad \overline{U}U = e\} \simeq \mathrm{U}(3^{\frac{d}{2}}). \tag{49}$$

If d is odd, then

$$\mathrm{U}C\ell^{\frac{1}{3}}_d = \{U \in C\ell^{\frac{1}{3}}_d : \ \overline{U}U = e\} \simeq \mathrm{U}(3^{\frac{d-1}{2}}) \times \mathrm{U}(3^{\frac{d-1}{2}}) \times \mathrm{U}(3^{\frac{d-1}{2}}). \tag{50}$$

The Lie group $\mathrm{SU}(3)$ and its Lie algebra $\mathfrak{su}(3)$ are widely used in physics to describe strong interactions in quantum chromodynamics. Presented realizations of unitary Lie groups and algebras can be useful in different applications of ternary and generalized Clifford algebras.

The 35th International Colloquium on Group Theoretical Methods in Physics (ICGTMP, Group35) will be held in **Cotonou, Benin, July 15 - 19, 2024.**

https://icgtmp.sciencesconf.org/. Registration deadline: **May 1, 2024.**

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PARALLEL SESSIONS

- Group theory, Lie groups, semigroups, Lie superalgebras, representation theory, and harmonic analysis (**organizers**: Melanija Mitrović, Serbia; Kinvi Kangni, Côte d'Ivoire)
- Probability, statistical models, infinite dimensional analysis and related topics (**organizers**: Youssef Ouknine, Marocco; Sophie Dabo, France; Simplice Dossou-Gbété, France)
- Discrete mathematics, graph theory, combinatorics and applications in theoretical physics (**organizers**: Remi Avohou, Benin; Audace Olory, Benin, and Eric Andriantiana, South Africa)
- (Super)integrable models, geometric mechanics, geometric methods and symmetries (**organizers**: Sonja Hohloch, Belgium; Partha Guha, United Arab Emirates; Cyriaque Atindogbé, Benin)
- Quantum systems, quantum mechanics, quantification techniques and coherent states (**organizers**: Jean-Piere Gazeau, France; and Laure Gouba, Italy)
- Supersymmetry, noncommutative geometry, string theory, quantum gravity, and field theories (**organizers**: Joseph Ben Geloun, France; and Dine Ousmane Samary, Benin)
- Clifford algebras, Clifford analysis and applications (**organizers**: Dmitry Shirokov, Russia; and ...)
- Hopf algebras, quantum groups, and K-theory
- Hom-algebras, Lie groupoids, Lie algebroids and nonassociative algebras (**organizers**: Abdenacer Makhlouf, France; and)
- Lie algebraic aspects in mathematical physics (**organizers**: Chengming Bai, China; Yun Gao, Canada)
- Signal theory, quantum formalism, wavelets and related topics (organizers: Romain Murenzi, Italy; Bruno Torresani, France)
- Operator theory, differential/difference equations, orthogonal polynomials, and special functions (**organizers**: Juma Shabani, Burundi; Mama Foupouagnigni, Cameroon)
- Modeling, Analysis, control theory, partial differential equations and cellular automata, mathematical biology/epidemiology, environment(**organizers**: A. Samed Bernoussi, Morocco; Guy Dègla, Benin)
- Fluid mechanics, and related topics (**organizers**: Jean-Chabi Orou, Benin;)
- Nonlinear Optics, functional materials and related topics (**organizers**: Bouchta Sahraoui, France; and)
- Random matrix theory and related topics
- Adinkra symbols in mathematical physics

The ICGTMP was founded in 1972. It is led by a Standing Committee, which helps select winners for the three major awards presented at the conference: the Wigner Medal (1978–2018), the Hermann Weyl Prize (since 2002) and the Weyl-Wigner Award (since 2022).

Thank you for your attention!