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On grade automorphism and unitary groups in ternary Clifford algebras

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Abstract

In this talk, we discuss the operation of Hermitian transpose in ternary Clifford algebras [1, 2, 3] and the corresponding unitary group SU(3), which is important for physical applications. The explicit basis of the corresponding Lie algebra $\mathfrak{su}(3)$ is presented. We introduce a grade automorphism in ternary Clifford algebras and prove a number of its properties. The results are generalized to the case of Generalized Clifford algebras and the corresponding unitary groups of other dimensions.

Below we present an extended abstract of the talk. The operation (14) is considered for the first time. Theorems 2.1, 2.2, 3.1, 3.2 are new. Realizations of determinant and Lie group SU(3) (see Sections 4 and 5) in ternary Clifford algebra are new. More details and proofs of theorems will be presented in [5].

1 Ternary and generalized Clifford algebras

Let us consider a ternary Clifford algebra with two generators $C\ell_2^{\frac{1}{3}}$ [1]. The generators satisfy

$$e_1^3 = e_2^3 = 1, \tag{1}$$

$$e_1 e_2 = \omega e_2 e_1, \qquad \omega = e^{\frac{2\pi i}{3}}. \tag{2}$$

An arbitrary element $U \in C\ell_2^{\frac{1}{3}}$ has the form

$$U = \sum_{j,k=0}^{2} u_{jk} e_1^j e_2^k = u_{00} e + u_{10} e_1 + u_{01} e_2 + u_{20} e_1^2 + u_{02} e_2^2 + u_{11} e_1 e_2 \tag{3}$$

$$+ u_{21} e_1^2 e_2 + u_{12} e_1 e_2^2 + u_{22} e_1^2 e_2^2, \qquad u_{jk} \in \mathbb{C}. \tag{4}$$

1st \ 2nd	e_1	e_2	e_1^2	e_2^2	e_1e_2	$e_1^2e_2$	$e_1e_2^2$	$e_1^2e_2^2$
e_1	e_1^2	e_1e_2	e	$e_1e_2^2$	$e_1^2e_2$	e_2	$e_1^2e_2^2$	e_2^2
e_2	$\omega^2e_1e_2$	e_2^2	$\omega e_1^2e_2$	e	$\omega^2e_1e_2^2$	$\omega e_1^2e_2^2$	ω^2e_1	$\omega^2e_1^2$
e_1^2	e	$e_1^2e_2$	e_1	$e_1^2e_2^2$	e_2	e_1e_2	e_2^2	$e_1e_2^2$
e_2^2	$\omega e_1e_2^2$	e	$\omega^2e_1^2e_2^2$	e_2	ωe_1	$\omega^2e_1^2$	ωe_1e_2	$\omega e_1^2e_2$
e_1e_2	$\omega^2e_1^2e_2$	$e_1e_2^2$	ωe_2	e_1	$\omega^2e_1^2e_2^2$	ωe_2^2	$\omega^2e_1^2$	ωe
$e_1^2e_2$	ω^2e_2	$e_1^2e_2^2$	ωe_1e_2	e_1^2	$\omega^2e_2^2$	$\omega e_1e_2^2$	ω^2e	ωe_1
$e_1e_2^2$	$\omega e_1^2e_2^2$	e_1	$\omega^2e_2^2$	e_1e_2	ωe_1^2	ω^2	$\omega e_1^2e_2$	ωe_2
$e_1^2e_2^2$	ωe_2^2	e_1^2	$\omega^2e_1e_2^2$	$e_1^2e_2$	ω	ω^2e_1	ωe_2	ωe_1e_2

Tab. 1: Multiplication table in $C\ell_2^{\frac{1}{3}}$.

The multiplication table is the following (see Table 1).
Let us consider the following explicit matrix representation (isomorphism)

$$\beta : C\ell_2^{\frac{1}{3}} \to \mathrm{Mat}(3, \mathbb{C}).$$

We use the following matrices

$$\beta(e) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \beta(e_1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \beta(e_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix},$$

$$\beta(e_1^2) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \beta(e_2^2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{bmatrix},$$

$$\beta(e_{12}) = \begin{bmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{bmatrix}, \beta(e_1^2e_2) = \begin{bmatrix} 0 & 0 & \omega^2 \\ 1 & 0 & 0 \\ 0 & \omega & 0 \end{bmatrix},$$

$$\beta(e_1e_2^2) = \begin{bmatrix} 0 & \omega^2 & 0 \\ 0 & 0 & \omega \\ 1 & 0 & 0 \end{bmatrix}, \beta(e_1^2e_2^2) = \begin{bmatrix} 0 & 0 & \omega \\ 1 & 0 & 0 \\ 0 & \omega^2 & 0 \end{bmatrix}.$$

For an arbitrary $U \in C\ell_2^{\frac{1}{3}}$, we get

$$\beta(U) = \begin{bmatrix} u_{00} + u_{01} + u_{02} & u_{10} + \omega u_{11} + \omega^2 u_{22} & u_{20} + \omega u_{22} + \omega^2 u_{21} \\ u_{20} + u_{21} + u_{22} & u_{00} + \omega u_{01} + \omega^2 u_{02} & u_{10} + \omega u_{12} + \omega^2 u_{11} \\ u_{10} + u_{11} + u_{12} & u_{20} + \omega u_{21} + \omega^2 u_{22} & u_{00} + \omega u_{02} + \omega^2 u_{01} \end{bmatrix}. \quad (5)$$

Let us consider the generalized Clifford algebra $C\ell_n^{\frac{1}{m}}$ with n generators that satisfy

$$e_j^m = e, \qquad j = 1, \ldots, n, \quad (6)$$

$$e_i e_j = \omega e_j e_i, \qquad i < j, \qquad \omega = e^{\frac{2\pi i}{m}}. \quad (7)$$

We have $\dim(C\ell_n^{\frac{1}{m}}) = m^n$. An arbitrary element $U \in C\ell_n^{\frac{1}{m}}$ has the form

$$U = \sum_{j_1,\ldots,j_n=0}^{m-1} u_{j_1\ldots j_n} e_1^{j_1} \cdots e_n^{j_n} \tag{8}$$
$$= u_{0\ldots0}e + u_{10\ldots0}e_1 + \cdots + u_{0\ldots01}e_n + \cdots + u_{m-1\ldots m-1}e_1^{m-1} \cdots e_n^{m-1}$$

We have the following particular cases. In the case $m = 2$, we obtain standard (complex) Clifford algebra $C\ell_n$ [4]. In the case $m = 3$, we obtain ternary Clifford algebra $C\ell_n^{\frac{1}{3}}$.

2 Grade authomorphism in $C\ell_n^{\frac{1}{3}}$

Let us consider the subspace $C\ell_n^{\frac{1}{3},k}$ of grade k with the basis elements that are products of k generators. We have

$$C\ell_n^{\frac{1}{3}} = \bigoplus_{k=0}^{2n} C\ell_n^{\frac{1}{3},k}. \tag{9}$$

It is easily verified that

$$C\ell_n^{\frac{1}{3},k} C\ell_n^{\frac{1}{3},j} \subseteq \bigoplus_{s=0}^{[\frac{k+j}{3}]} C\ell_n^{\frac{1}{3},k+j-3s} \tag{10}$$

We have

$$C\ell_n^{\frac{1}{3}} = C\ell_n^{\frac{1}{3},(0)} \oplus C\ell_n^{\frac{1}{3},(1)} \oplus C\ell_n^{\frac{1}{3},(2)}, \qquad C\ell_n^{\frac{1}{3},(k)} := \bigoplus_{j=k \bmod 3} C\ell_n^{\frac{1}{3},j}. \tag{11}$$

We have $\mathbb{Z}_3$-grading (see [1]):

$$C\ell_n^{\frac{1}{3},(k)} C\ell_n^{\frac{1}{3},(j)} \subseteq C\ell_n^{\frac{1}{3},(k+j) \bmod 3} \tag{12}$$

We use the notation

$$U = \sum_{j=0}^{2} U_{(j)}, \qquad U_{(j)} \in C\ell_n^{\frac{1}{3},(k)}. \tag{13}$$

Let us consider the following operation (*grade authomorphism*)

$$\widehat{U} := U_{(0)} + \omega U_{(1)} + \omega^2 U_{(2)}, \qquad U \in C\ell_n^{\frac{1}{3}}. \tag{14}$$

We have

$$C\ell_n^{\frac{1}{3},(k)} = \{U \in C\ell_n^{\frac{1}{3}} : \widehat{U} = \omega^k U\}, \qquad k = 0, 1, 2. \tag{15}$$

Theorem 2.1: The operation (14) in $C\ell_n^{\frac{1}{3}}$ has the properties

$$\widehat{\widehat{\widehat{U}}} = U, \qquad (\alpha U + \beta V)\widehat{\ } = \alpha\widehat{U} + \beta\widehat{V}, \qquad \widehat{UV} = \widehat{U}\widehat{V}. \tag{16}$$

Theorem 2.2: The operation (14) in $C\ell_n^{\frac{1}{3}}$ is an inner authomorphism.

In the particular case $C\ell_2^{\frac{1}{3}}$, we have

$$\widehat{U} = T^{-1}UT, \qquad T = e_1^2 e_2. \tag{17}$$

3 Lie groups

Theorem 3.1: The center of $C\ell_n^{\frac{1}{m}}$ has the form

$$\mathrm{cen}(C\ell_n^{\frac{1}{m}}) = \{\lambda e, \qquad \lambda \in \mathbb{C}\} \tag{18}$$

in the case of even n and

$$\mathrm{cen}(C\ell_n^{\frac{1}{m}}) = \{\lambda e + \alpha e_1^1 e_2^{m-1} e_3^1 \cdots e_n^1 + \beta e_1^2 e_2^{m-2} e_3^2 \cdots e_n^2 + \cdots \tag{19}$$
$$+\phi e_1^{m-1} e_2^1 e_3^{m-1} \cdots e_n^{m-1}, \qquad \lambda, \alpha, \beta, \ldots, \phi \in \mathbb{C}\} \tag{20}$$

in the case of odd n.

The particular case of this theorem for ternary Clifford algebras (the case $m = 3$) is considered in [1]:

$$\mathrm{cen}(C\ell_n^{\frac{1}{3}}) = \{\lambda e + \alpha e_1^1 e_2^2 e_3^1 \cdots e_n^1 + \beta e_1^2 e_2^1 e_3^2 \cdots e_n^2, \quad \lambda, \alpha, \beta \in \mathbb{C}\}. \tag{21}$$

Let us denote the group of all invertible elements of the center of $C\ell_n^{\frac{1}{m}}$ by $Z^\times$. Let us consider the groups

$$\Gamma^{(k)} := \{T \in C\ell_n^{\frac{1}{m}} : T^{-1} C\ell_n^{\frac{1}{m},(k)} T \subset C\ell_n^{\frac{1}{m},(k)}\}, \quad k = 0, 1, \ldots, m-1;$$
$$P := Z^\times \bigcup_{k=0}^{m-1} C\ell_n^{\frac{1}{m},(k)}. \tag{22}$$

Lemma 3.1: Suppose $T \in C\ell_n^{\frac{1}{m}}$ is invertible. If $T \in C\ell_n^{\frac{1}{m},(k)}$, $k = 0, 1, \ldots, m-1$, then $T^{-1} \in C\ell_n^{\frac{1}{m},(-k) \mod m}$.

Theorem 3.2: The following Lie groups coincide:

$$\Gamma^{(k)} = P, \qquad k = 0, 1, \ldots, m-1. \tag{23}$$

4 The determinant and inverse

Let us introduce the notion of determinant of multivector $U \in C\ell_2^{\frac{1}{3}}$:

$$\det(U) := \det(\beta(U))$$

for an arbitrary matrix representation of minimal dimension

$$\beta : C\ell_2^{\frac{1}{3}} \to \mathrm{Mat}(3, \mathbb{C}). \tag{24}$$

For (4), we get

$$\begin{aligned}\det(U) &= u_{00}^3 + u_{10}^3 + u_{01}^3 + u_{20}^3 + u_{02}^3 + u_{11}^3 + u_{21}^3 + u_{12}^3 + u_{22}^3 \\ &-3(u_{00}u_{01}u_{02} + u_{10}u_{11}u_{12} + u_{00}u_{10}u_{20} + u_{01}u_{11}u_{21} + u_{02}u_{12}u_{22} \\ &+u_{20}u_{21}u_{22}) - 3\omega(u_{01}u_{12}u_{20} + u_{02}u_{10}u_{21} + u_{00}u_{11}u_{22}) \\ &-3\omega^2(u_{02}u_{11}u_{20} + u_{00}u_{12}u_{21} + u_{01}u_{10}u_{22})\end{aligned} \tag{25}$$

Let us introduce the operation

$$\overline{U} := U_0 - \sum_{k=1}^{4} U_k = 2U_0 - U, \qquad U \in C\ell_2^{\frac{1}{3}}, \qquad U_k \in C\ell_2^{\frac{1}{3},k}.$$

For the coefficients $C_{(k)}$, $k = 1, 2, 3$ of characteristic polynomial of an arbitrary multivector $U \in C\ell_2^{\frac{1}{3}}$, we get:

$$U_{(1)} = U, \qquad C_{(1)} = 3\langle U \rangle_0 = \frac{3}{2}(U + \overline{U}), \tag{26}$$

$$U_{(2)} = U(U - C_{(1)}) = -\frac{1}{2}U(U + 3\overline{U}), \tag{27}$$

$$C_{(2)} = \frac{3}{2}\langle U_{(2)} \rangle_0 = -\frac{3}{8}U^2 - \frac{3}{8}\overline{U^2} - \frac{9}{8}\overline{U} - \frac{9}{8}U, \tag{28}$$

$$\det(U) = -U_{(3)} = -C_{(3)} = -U(U - C_{(2)}) \tag{29}$$

$$= -U(\frac{17}{8}U + \frac{3}{8}U^2 + \frac{3}{8}\overline{U^2} + \frac{9}{8}\overline{U}). \tag{30}$$

In the case $\det(U) \neq 0$, we get an explicit formula for the inverse:

$$U^{-1} = -\frac{17U + 3U^2 + 3\overline{U^2} + 9\overline{U}}{8\det(U)}. \tag{31}$$

5 The Lie group SU(3) and the Lie algebra su(3)

Let us consider the following operation of Hermitian conjugation [1]:

$$U^\dagger := U|_{u_A \to \bar{u}_A,\, e_A \to e_A^{-1}}, \qquad \forall U \in C\ell_2^{\frac{1}{3}}. \tag{32}$$

For the matrix representation β of minimal dimension (24), we have

$$(\beta(U))^\dagger = \beta(U^\dagger).$$

Let us consider the Lie group

$$G_2^{\frac{1}{3}} := \{U \in C\ell_2^{\frac{1}{3}} : \quad U^\dagger U = e, \quad \det(U) = 1\}, \tag{33}$$

where det is defined by (30).

The Lie group $G_2^{\frac{1}{3}}$ is isomorphic to SU(3).

The corresponding Lie algebra has the following form

$$\mathfrak{g}_2^{\frac{1}{3}} := \{U \in C\ell_2^{\frac{1}{3}} : \quad U^\dagger = -U, \quad \mathrm{tr(U)} = 0\} \simeq \mathfrak{su}(3). \tag{34}$$

Lemma 5.1: The basis of $\mathfrak{su}(3)$ is

$$ie_1, \quad ie_2, \quad ie_1^2, \quad ie_2^2, \quad ie_1e_2, \quad ie_1^2e_2, \quad ie_1e_2^2, \quad ie_1^2e_2^2. \tag{35}$$

Another basis of $\mathfrak{su}(3)$ is constructed using well-known Gell–Mann matrices (with multiplication by imaginary unit i). The relation between these two bases is presented in [2].

In the case of generalized Clifford algebras $C\ell_n^{\frac{1}{m}}$, we have the following isomorphisms with unitary groups.

If n is even, then

$$G_n^{\frac{1}{m}} = \{U \in C\ell_n^{\frac{1}{m}} : \quad U^\dagger U = e\} \simeq \mathrm{U}(\mathrm{m}^{\frac{\mathrm{n}}{2}}) \tag{36}$$

If n is odd, then

$$G_n^{\frac{1}{m}} = \{U \in C\ell_n^{\frac{1}{m}} : \quad U^\dagger U = e\} \simeq \mathrm{U}(\mathrm{m}^{\frac{\mathrm{n}-1}{2}}) \times \mathrm{U}(\mathrm{m}^{\frac{\mathrm{n}-1}{2}}) \times \cdots \times \mathrm{U}(\mathrm{m}^{\frac{\mathrm{n}-1}{2}}), \tag{37}$$

where we have m multipliers.

The Lie group SU(3) and its Lie algebra $\mathfrak{su}(3)$ are widely used in physics to describe strong interactions in quantum chromodynamics. Presented realizations of unitary Lie groups and algebras can be useful in different application of ternary and generalized Clifford algebras in physics and computer science.

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