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S-stable tournament solutions: review and new results

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Alternatives and comparisons

A – the *general set* of alternatives.

X – the *menu*: $X \subseteq A \wedge |X| < \infty$.

R – results of binary comparisons, $R \subseteq A \times A$. R reveals and represents chooser's preferences.

P – asymmetric part of R, $P \subseteq R$. P represents strict preferences.

R is presumed to be complete: $\forall\, x, y \in A$, $xRy \vee yRx$. That is, R and P are dual.

If there are no indifferences, then (X, P) – (*proper*) *tournament*.

If there are indifferences, then (X, P) – *weak tournament*.

$R|_X$ denotes restriction of R onto X, $R|_X = R \cap X \times X$

Optimal choices

The choice is a partition of X into two subsets - the choice set S and the set $X \setminus S$ of rejected alternatives.

Thus, choices are represented by a correspondence $S(X, P)$: $2^A \times 2^{A \times A} \to 2^A$

A rational chooser should optimize. Optimization is understood as *maximization of preferences*.

The alternative that P-dominates any other alternative in X is called the *Condorcet winner*.

An alternative that R-dominates any other alternative in X is called *a maximal element* (of $R|_X$).

$CW(X, P)$ - the set of Condorcet winners.

$MAX(X, P)$ - the set of maximal elements of $R|_X$.

A maximal element of the preference relation is presumed to be the best choice for the chooser,

and a maximal element which is the Condorcet winner is presumed to be the only best choice.

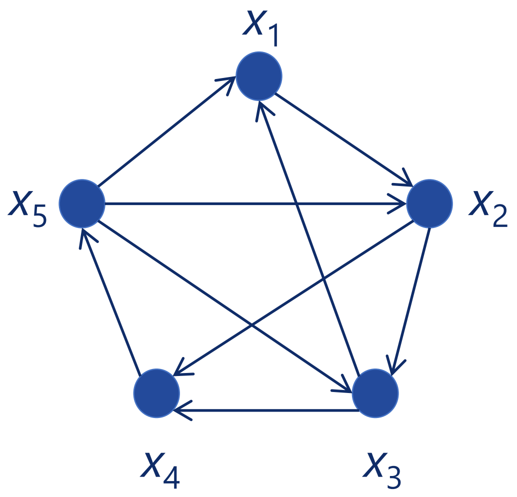
Condorcet consistency: $MAX(X, P) \subseteq S(X, P) \wedge CW(X, P) \neq \varnothing \Rightarrow S(X, P) = CW(X, P)$.

Tournament solutions

A *tournament solution S* is a choice correspondence that has the following properties:

0. *Locality*: $S(X, P)=S(P|_X) \subseteq X$
1. *Nonemptiness*: $\forall X \neq \varnothing, \forall P, S(X, P) \neq \varnothing$;
2. *Neutrality*: permutation of alternatives' names and choice commute;
3. *Condorcet consistency*:

$$MAX(X, P) \subseteq S(X, P) \wedge (CW(X, P) \neq \varnothing \Rightarrow S(X, P)=CW(X, P)).$$



Tournament digraph

Top cycle. Uncovered set. Minimal covering set

A subset Y of menu X is a *dominant set* if $\forall\, x \in Y, \forall\, y \in X \setminus Y, xPy$.

$TC(X, P)$ - the (unique) minimal dominant set (Top cycle).

Alternative x *covers* alternative y in X if $xPy \wedge \forall\, z \in X, yPz \Rightarrow xPz$.

$UC(X, P)$ - the set of uncovered (in X) alternatives (Uncovered set).

A subset Y of menu X is a *covering set* if the following two conditions hold:

1. $UC(Y, P)=Y$;
2. $\forall\, x \in X \setminus Y, x \notin UC(Y \cup \{x\}, P)$.

$MC(X, P)$ - the (unique) minimal covering set.

Stability external and internal. *S*-stability

A subset Y of menu X is an *internally stable* set if $\forall\ x, y \in Y, \neg yPx$.

A subset Y of menu X is an *externally stable* set if $\forall\ x \in X \backslash Y, \exists\ y \in Y : yPx$.

1. $MAX(Y, P)=Y$;
2. $\forall\ x \in X \backslash Y, x \notin MAX(Y \cup \{x\}, P)$.

A subset Y of menu X is an *internally S-stable* set if $S(Y, P)=Y$.

A subset Y of menu X is an *externally S-stable* set if $\forall\ x \in X \backslash Y, x \notin S(Y \cup \{x\}, P)$.

A von Neumann-Morgenstern stable set is an internally and externally *MAX-stable set*.

A covering set is an internally and externally *UC-stable set*.

The minimal externally *UC*-stable set is unique and also internally *UC*-stable.

Therefore, the minimal externally *UC*-stable set is the minimal covering set *MC*.

Thus, one may define the covering set as only externally *UC*-stable and obtain the same *MC*.

For any tournament solution *S*(*X*, *P*) there is another tournament solution $\hat{S}$:

$\hat{S}$(*X*, *P*) – the union of minimal externally *S*-stable sets.

$$MC=\widehat{UC}$$

An alternative that *P*-dominates some other alternative in *X* is called a *Condorcet non-looser*.

CNL(*X*, *P*) - the set of Condorcet non-loosers.

The minimal externally *CNL*-stable set is the minimal dominant set *TC.* It is unique and also internally *CNL*-stable.

$$TC=\widehat{CNL}$$

Self-stability

A tournament solution S is *self-stable*

if $S(X, P)$ is the unique minimal internally and externally *S-stable* set in X.

If S is self-stable then $\hat{S}=S$.

MC and TC are self-stable. Therefore, $\widehat{MC}=MC$, $\widehat{TC}=TC$.

Theorem (Brand and Harrenstein, 2011): S is self-stable

if and only if

$Z \subseteq X \cap Y$ and $Z \neq \emptyset$ imply $S(X, P)=S(Y, P)=Z \Leftrightarrow S(X \cup Y, P)=Z$ **(Stability)**.

Stability and properties related to stability

Stability: $Z \subseteq X \cap Y$ and $Z \neq \varnothing$ imply $S(X)=S(Y)=Z \Leftrightarrow S(X \cup Y)=Z$.

- $\hat{\alpha}$***-property*** (*the generalized Nash independence of irrelevant alternatives or the Outcast property or the Strong Superset property*):

$$S(X)=S(Y)=Z \Leftarrow (S(X \cup Y)=Z \wedge Z \subseteq X \wedge Z \subseteq Y).$$

- $\hat{\gamma}$***-property***:

$$S(X)=S(Y)=Z \Rightarrow S(X \cup Y)=Z.$$

- ***Idempotence***: $\forall X,\ S(S(X))=S(X)$.
- ***The Aïzerman condition***: $\forall X, \forall Y,\ S(X) \subseteq Y \subseteq X \Rightarrow S(Y) \subseteq S(X)$.

$$\textit{Outcast} \Leftrightarrow \textit{Idempotence} \wedge \textit{the Aïzerman condition}$$

Self-stability of $\widehat{S}$

Is the union of minimal externally *S*-stable sets (self-)stable?

Not always!

BA(*X*, *P*) - the *Banks set* (the union of maximal elements of all maximal chains).

$\widehat{BA}$ is not stable.

Theorem: $\hat{S}$ is self-stable

if and only if

the minimal externally *S*-stable set is uniquely defined.

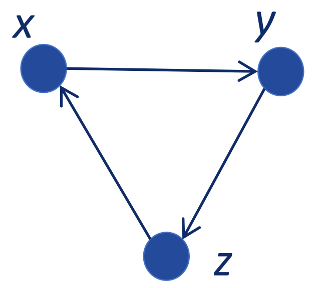
Since minimal externally *UC*- and *CNL*-stable sets are uniquely defined, corresponding solutions *MC*=$\widehat{UC}$ and *TC*=$\widehat{CNL}$ are (self-)stable.

Stability of $\widehat{MAX}$

Let us consider the union of minimal externally stable sets *ES*. By definition, *ES*= $\widehat{MAX}$.

Is the minimal externally stable set uniquely defined?

No!



Minimal externally stable sets are {*x, y*}, {*x, z*}, {*y, z*}.

Does the union of minimal externally stable sets *ES* satisfy stability?

Yes!

Why?

$\hat{S}$ was originally defined for tournament solutions *S.*

But *MAX*(*X*, *R*) – is not a tournament solution, since it doesn't satisfy *Nonemptiness*!

Versions of *ES* in weak tournaments

Version 1. If $\forall x \in X \setminus Y$, $\exists y \in Y$: yPx then Y is a P-externally stable set.

The *union of minimal P-externally stable sets ES*= $\widehat{MAX}$. *ES* satisfies stability.

Version 2. If $\forall x \in X \setminus Y$, $\exists y \in Y$: yRx then Y is a R-externally stable set.

The *union of minimal R-externally stable sets RES*= $\widehat{CW}$.

RES satisfies $S(X) \subseteq Y \subseteq X \Rightarrow S(Y)=S(Y)$ (the Outcast property) and $S(X)=S(Y) \Rightarrow S(X) \subseteq S(X \cup Y)$.

But *RES* violates stability since it is possible that $S(X)=S(Y) \wedge S(X) \subset S(X \cup Y)$

Version 3. If $\forall x \in X \setminus Y$, $(\exists y \in Y: yPx) \vee (\forall y \in Y: yRx)$ then Y is *weakly stable* (Aleskerov & Kurbanov, 1999).

Let *PW* select all *partial winners*, $PW(X, P)=\{x \in X \mid (\forall y \in X, xRy) \wedge (\exists y \in X, xPy)\}$.

The *union of minimal weakly stable sets WS*= $\widehat{PW}$.

WS violates the Outcast property, so it is not stable.

Thank you!

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