National Research University
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Decision Choice and Analysis
Laboratory (DeCAn Lab)

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Choosing optimal sets: stable tournament solutions and their extensions

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Alternatives, comparisons, choices

A – the *general set* of alternatives.

X – a *menu*: $X \subseteq A \wedge |X|<\infty$.

R – results of binary comparisons, $R \subseteq A\times A$.

P – the asymmetric part of R (strict preferences), $P \subseteq R$: $(x, y) \in P \Leftrightarrow ((x, y) \in R \wedge (y, x) \notin R)$.

R is presumed to be complete: $\forall\, x \in A, \forall\, y \in A, (x, y) \in R \vee (y, x) \in R$.

Consequently, R and P are dual.

(X, P) – *abstract game or weak* *tournament*.

If $P|_X = P\cap X\times X$ is connex, $\forall\, x \in X, \forall\, y \in X \wedge y \neq x, (x, y) \in P \vee (y, x) \in P$, then

(X, P) – (*proper*) *tournament*.

Tournament solutions

A *tournament solution* S is a choice correspondence

$$S(X, P): 2^A \times 2^{A\times A} \rightarrow 2^A$$

that has the following properties:

0. Locality: $S(X, P)=S(P|_X) \subseteq X$

1. Nonemptiness: $\forall X\neq\varnothing, \forall P, \ S(X, P)\neq\varnothing$;

2. Neutrality: permutation of alternatives' names and choice commute;

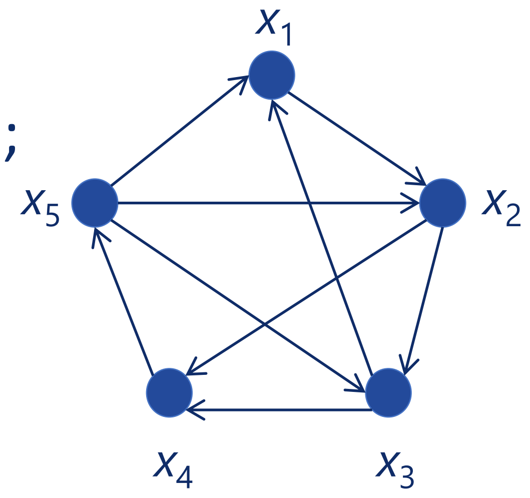
3. Condorcet consistency:

a) $MAX(P|_X) \subseteq S(X, P)$

b) $(\exists\ cw \in X: \forall\ x \in X, cwPx) \Rightarrow S(X, R)=\{cw\}$.

	x_1	x_2	x_3	x_4	x_5
x_1	0	1	0	0	0
x_2	0	0	1	1	0
x_3	1	0	0	1	0
x_4	0	0	0	0	1
x_5	1	1	1	0	0

Tournament matrix **T**



Tournament digraph

Properties

- ***P-monotonicity*** (*monotonicity with respect to preferences P*)

  If $P_1|_{X\setminus\{x\}}=P_2|_{X\setminus\{x\}} \wedge \forall y \in X\setminus\{x\}, (xP_1y \Rightarrow xP_2y) \wedge (xR_1y \Rightarrow xR_2y)$

  then $x \in S(X, P_1) \Rightarrow x \in S(X, P_2)$.

- ***Stability***

  $Z \subseteq X \cap Y$ and $Z \neq \varnothing$ imply $S(X, P)=S(Y, P)=Z \Leftrightarrow S(X \cup Y, P)=Z$.

- ***Computational simplicity***

  There is a polynomial algorithm for computing *S*.

Properties related to stability

Stability: $Z \subseteq X \cap Y$ and $Z \neq \emptyset$ imply $S(X, P)=S(Y, P)=Z \Leftrightarrow S(X \cup Y, P)=Z$.

- ***$\hat{\alpha}$-property*** (*Outcast, SSP, generalized Nash independence of irrelevant alternatives*):

$$S(X, P)=S(Y, P)=Z \Leftarrow (S(X \cup Y, P)=Z \wedge Z \subseteq X \wedge Z \subseteq Y).$$

- ***$\hat{\gamma}$-property***: $S(X, P)=S(Y, P)=Z \Rightarrow S(X \cup Y, P)=Z$.
- ***Idempotence***: $\forall X, S(S(X, P), P)=S(X, P)$.

- ***The Aïzerman condition***: $\forall X, \forall Y, S(X, P) \subseteq Y \subseteq X \Rightarrow S(Y, P) \subseteq S(X, P)$.
- ***Independence of irrelevant comparisons*** (*independence of losers*): If P_1 and P_2 are such that $\forall x \in S(X, P_1), \forall y \in X, ((xP_1y \Leftrightarrow xP_2y) \wedge (xR_1y \Leftrightarrow xR_2y))$ then $S(X, P_2)=S(X, P_1)$.

Outcast $\Leftrightarrow$ *Idempotence* $\wedge$ *the Aïzerman condition*

Outcast $\wedge$ *P-monotonicity* $\Rightarrow$ *Independence of irrelevant comparisons*

Top cycle *TC*. The union of minimal externally stable sets *ES*

A subset *Y* of menu *X* is a *dominant set* if $\forall\, x \in X \backslash Y,\ \forall\, y \in Y,\ yPx$.

TC(*X*, *P*) - the (unique) minimal dominant set (*Top cycle*).
(Schwartz 1970, 1972, Good 1971, Smith 1973)

A subset *Y* of menu *X* is an *externally stable set* if $\forall\, x \in X \backslash Y,\ \exists\, y \in Y: yPx$.
ES(*X*, *P*) - the union of minimal externally stable sets.
(Wilhelm 1977, Wuffl, Feld, Owen, Grofman 1989, Subochev 2008)

Bipartisan set *BP*

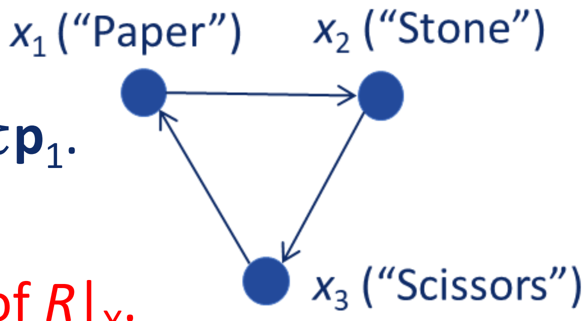
Comparison function: $g(x_1, x_2)=1 \Leftrightarrow x_1Px_2$, $g(x_1, x_2)=-1 \Leftrightarrow x_2Px_1$, otherwise $g(x_1, x_2)=0$.

Since matrix $\mathbf{G} = \|g(x_i, x_j)\|$ is skew-symmetric,

formula $\mathbf{p}_1\mathbf{G}\mathbf{p}_2$ defines a binary relation on the set of lotteries: $\mathbf{p}_1\mathbf{G}\mathbf{p}_2 \geq 0 \Leftrightarrow \mathbf{p}_1 \gtrsim \mathbf{p}_1$.

If $\mathbf{p}_0\mathbf{G}\mathbf{p} \geq 0$ for all $\mathbf{p}$ then $\mathbf{p}_0$ is a *maximal lottery*.

The set $\{x\}$ is the support of a maximal lottery on $X \Leftrightarrow x$ is a maximal element of $R|_X$.



Tournament game – "Paper, Scissors, Stone"

Example: The Condorcet cycle. $X=\{x_1, x_2, x_3\}$, $R|_X=\{(x_1, x_2), (x_1, x_2), (x_1, x_2)\}$.

Maximal lottery $\mathbf{p}_{max}$= (1/3, 1/3, 1/3).

Note that $\mathbf{p}_{max}$ is an eigenvector of $\mathbf{G}$ with the eigenvalue 0, and $\mathbf{p}\mathbf{G}\mathbf{p}_{max}=0$ for all $\mathbf{p}$.

	x_1	x_2	x_3
x_1	0	1	-1
x_2	-1	0	1
x_3	1	-1	0

Matrix **G**

Bipartisan set BP (Laffond, Laslier, Le Breton, 1993)

of a (proper) tournament (X, P) is the support of the (unique) maximal lottery.

Minimal covering set MC

Alternative x *covers* alternative y in X if $xPy \wedge \forall z \in X, yPz \Rightarrow xPz$.

UC(X, P) - the set of uncovered (in X) alternatives (*Uncovered set*).

Covering set

Version 1 (Dutta, 1988) $Y \subseteq X$ is a *covering* set in $X \Leftrightarrow UC(Y) = Y$ and $\forall x \in X \setminus Y, x \notin UC(Y \cup \{x\})$.
Version 2 (Laslier, 1997; Brandt, 2011) $Y \subseteq X$ is a *covering* set in $X \Leftrightarrow \forall x \in X \setminus Y, x \notin UC(Y \cup \{x\})$.

Minimal covering set MC of a (proper) tournament (X, P) is the (unique) minimal covering set.

Properties of TC, ES, *BP and MC*

	TC	*ES*	*BP*	*MC*
Monotonicity	Yes	Yes	Yes	Yes
Stability	Yes	Yes	Yes	Yes
$\hat{\alpha}$-property (Outcast)	Yes	Yes	Yes	Yes
Idempotence	Yes	Yes	Yes	Yes
Aizerman-Aleskerov property	Yes	Yes	Yes	Yes
Independence of irrelevant comparisons	Yes	Yes	Yes	Yes
$\hat{\gamma}$-property	Yes	Yes	Yes	Yes
Computational simplicity	Yes	Yes	Yes	Yes
Uniqueness	Yes	No	Yes	Yes

- ***Uniqueness***

  There is just one “best” (minimal “good”) subset in each menu.

The conservative extension (Brandt, Brill, Harrenstein 2014, 2018)

A proper tournament (X, T) is called ***orientation*** of a weak tournament (X, P) if $P \subseteq T$.

For a tournament solution $S(X, P)$, its ***conservative extension*** (denoted $[S]$) to weak tournaments is the choice correspondence $[S](X, P)$ defined thus:

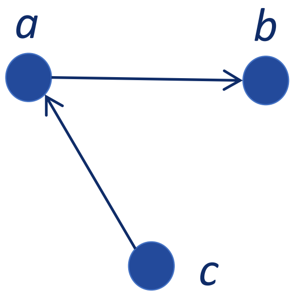
an alternative x from X belongs to $[S](X, P)$ if and only if

there is an ***orientation*** (X, T) of (X, P), such that x belong to $S(X, T)$.
That is, $[S](X, P)$ is the union of $S(X, T)$ over all orientations T of P.

Theorem: The conservative extension preserves properties of the original solution,

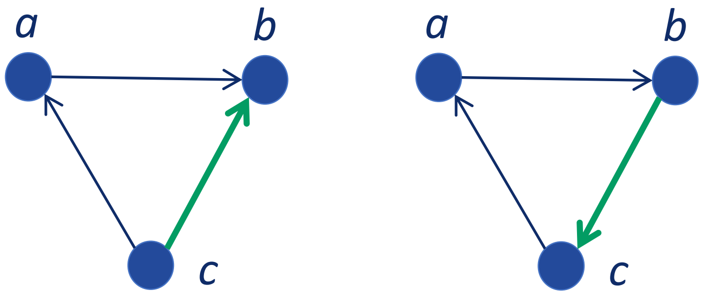
except computational simplicity and uniqueness.

Example



Weak tournament

$[BP]$=?



Orientations of the weak tournament

$BP=\{c\}$ $BP=\{a, b, c\}$

$[BP]=\{a, b, c\}$

Essential set *E*

1. The set of maximal lotteries is always nonempty.
2. When a tournament (*X*, *P*) is weak, there might be more than one maximal lottery.

Essential set E (Dutta, Laslier, 1999)

of a (weak) tournament (*X*, *P*) is the union of supports of all maximal lotteries.

Properties of *E*

	E
Monotonicity	Yes
Stability	Yes
$\hat{\alpha}$-property (Outcast)	Yes
Idempotence	Yes
Aizerman-Aleskerov property	Yes
Independence of irrelevant comparisons	Yes
$\hat{\gamma}$-property	Yes
Computational simplicity	Yes
Uniqueness	No

Versions of the covering relations and the uncovered sets in weak tournaments

The covering relations (Fishburn, 1977; Miller, 1980; McKelvey, 1986; Duggan, 2007, 2013)

$P(x)=\{y \in X | yPx\}$, $P^{-1}(x)=\{y \in X | xPy\}$, $P^{0}(x)=\{y \in X | \neg xPy \wedge \neg yRx\}$

- *upper section*, *lower section* and *horizon* of *x* in *X*, correspondingly.

The *covering* relation $C \subseteq X \times X$ is a strengthening of $P|_X$.

The *Miller* covering C_M: $xC_My \Leftrightarrow xPy \wedge P^{-1}(y) \subset P^{-1}(x)$. The *weak* Miller covering C_{WM}: $xC_{WM}y \Leftrightarrow P^{-1}(y) \subset P^{-1}(x)$.

The *Fishburn* covering C_F: $xC_Fy \Leftrightarrow xPy \wedge P(x) \subset P(y)$. The *weak* Fishburn covering C_{WF}: $xC_{WF}y \Leftrightarrow P(x) \subset P(y)$.

The *McKelvey* covering C_{McK}: $xC_{McK}y \Leftrightarrow xPy \wedge P^{-1}(y) \subset P^{-1}(x) \wedge P(x) \subset P(y)$.

The *weak* McKelvey covering C_{WMcK}: $xC_{WMcK}y \Leftrightarrow [P^{-1}(y) \subset P^{-1}(x) \wedge P(x) \subseteq P(y)] \vee [P^{-1}(y) \subseteq P^{-1}(x) \wedge P(x) \subset P(y)]$.

The *Duggan* (*deep*) covering C_D: $xC_Dy \Leftrightarrow P^{0}(y) \cup P^{-1}(y) \subset P^{-1}(x)$.

The *shallow* covering C_S: $xC_Sy \Leftrightarrow xPy \wedge P^{-1}(y) \subset P^{-1}(x) \cup P^{0}(y)$.

The set of all alternatives that are (weakly) covered in *X* by no alternative is *the (inner) uncovered set* of *X*.

The shallow, Miller, Fishburn, McKelvey and Duggan uncovered sets and their inner versions are denoted

UC_S UC_M UC_F UC_{McK} UC_D UC_{IM} UC_{IF} UC_{IMcK}.

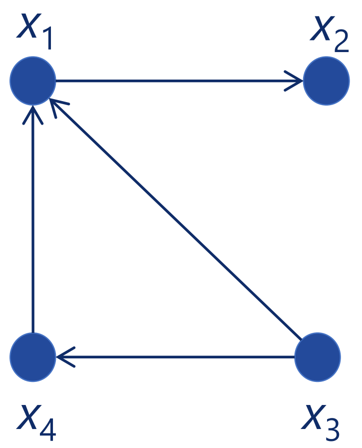
Versions of ***MC***

Covering set (Dutta, 1988)

- $\forall x \in X \setminus Y$, $x \notin UC(Y \cup \{x\})$, and
- $UC(Y) = Y$.

Covering set (Laslier, 1997)

- $\forall x \in X \setminus Y$, $x \notin UC(Y \cup \{x\})$.



$MC^*_{IMcK} = \emptyset$

$MC_{IMcK} = \{x_1, x_3, x_4\}$

***Minimal* V*-covering set* MC_V** (Brandt, 2011)
of a weak tournament (X, P) is the union of all minimal V-covering sets.

$Y \subseteq X$ is a **V-covering set** $\Leftrightarrow \forall x \in X \setminus Y$, $x \notin UC_V(Y \cup \{x\})$,
where **V**∈{S, M, IM, F, IF, McK, IMcK, D}.

Properties of the versions of *MC*

	MC_{S}	MC_{IM}	MC_{M}	MC_{IF}	MC_{F}	MC_{IMcK}	MC_{McK}	MC_{D}
Stability	No	No	No	No	No	No	Yes	Yes
$\hat{\alpha}$-property (Outcast)	No	No	No	No	No	No	Yes	Yes
Idempotence	No	No	No	No	Yes	No	Yes	Yes
Aïzerman property	No	No	No	No	No	No	Yes	Yes
$\hat{\gamma}$-property	No	No	No	No	No	No	Yes	Yes
Uniqueness	No	No	No	No	No	No	Yes	Yes

Versions of *TC* and *ES*

A nonempty subset Y of X is called

P-dominant	if	$\forall x \in X \backslash Y,$	$\forall y \in Y: yPx$
R-dominant	if	$\forall x \in X \backslash Y,$	$\forall y \in Y: yRx$
P-externally stable	if	$\forall x \in X \backslash Y,$	$\exists y \in Y: yPx$
R-externally stable	if	$\forall x \in X \backslash Y,$	$\exists y \in Y: yRx$
Weakly stable	if	$\forall x \in X \backslash Y,$	$(\exists y \in Y: yPx) \vee (\forall y \in Y, yRx)$

Tournament solutions: the union of all minimal

P-dominant sets *STC* a.k.a. the *strong top cycle* (Schwartz 1970, 1972)

R-dominant sets *WTC* a.k.a. the *weak top cycle* (Good 1971, Smith 1973)

P-externally stable sets *ES* (Wilhelm 1977, Wuffl, Feld, Owen, Grofman 1989, Subochev 2008)

R-externally stable sets *RES* (Aleskerov, Subochev 2009, 2013)

Weakly stable sets *WS* (Aleskerov, Kurbanov 1999)

Properties of the versions of *TC* and *ES*

	ES	*RES*	*WS*	*STC*	*WTC*
Monotonicity	Yes	Yes	Yes	Yes	Yes
Stability	Yes	No	No	No	Yes
$\hat{\alpha}$-property (Outcast)	Yes	Yes	No	No	Yes
Idempotence	Yes	Yes	No	Yes	Yes
Aïzerman property	Yes	Yes	No	No	Yes
Independence of irrelevant comparisons	Yes	Yes	No	No	Yes
$\hat{\gamma}$-property	Yes	No	No	No	Yes
Computational simplicity	Yes	Yes	?	Yes	Yes
Uniqueness	No	No	No	No	Yes

Theorem (Subochev 2024): $RES(X, P) = RES(Y, P) \Rightarrow RES(X, P) \subseteq RES(X \cup Y, P)$.

Thank you!

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