Квазигомогенная форма системы четырех уравнений динамики гетерогенных смесей газов и жидкостей, ее регуляризация и реализация

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1. A reduced system of PDEs for the dynamics of heterogeneous mixtures of stiffened gases and its further reduction

2. Regularized systems of PDEs for the dynamics of quasi-homogeneous mixtures of stiffened gases

3. Finite-difference schemes for the 1D regularized systems of PDes

4. Numerical experiments
   - Water-to-air shock tube problem

The four-equation model: no volume fraction Eq.

Common velocity, common temperature and equal pressures:

S. Le Martelot, R. Saurel, B. Nkonga, Towards the direct numerical simulation of nucleate boiling flows, Int. J. Multiphase Flow **66**, 62–78 (2014).

R. Saurel, P. Boivin, O. Le Métayer, A general formulation for cavitating, boiling and evaporating flows, Comput. Fluids **128**, 53–64 (2016).

Our paper containing the results of the report:

A. Zlotnik, T. Lomonosov. On a doubly reduced model for dynamics of heterogeneous mixtures of stiffened gases, its regularizations and their implementations // Chaos. 2023. Vol. 33. No. 11. Article 113128.

The four-equation system of PDEs for the heterogeneous one-velocity and one-temperature compressible binary mixture consists of the balance PDEs for the mass of components, total momentum and total energy

$$\partial_t(\alpha_k r_k) + \operatorname{div}(\alpha_k r_k \boldsymbol{u}) = 0, \quad k = 1,2, \tag{1}$$

$$\partial_t(\rho \boldsymbol{u}) + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = \operatorname{div} \Pi^{NS} + \rho \boldsymbol{f}, \tag{2}$$

$$\partial_t(\tfrac{1}{2}\rho|\boldsymbol{u}|^2 + \rho\varepsilon) + \operatorname{div}\left((\tfrac{1}{2}\rho|\boldsymbol{u}|^2 + \rho\varepsilon + p)\boldsymbol{u}\right) = \operatorname{div}(-\boldsymbol{q}^F + \Pi^{NS}\boldsymbol{u}) + \rho \boldsymbol{u} \cdot \boldsymbol{f}. \tag{3}$$

Here the main sought functions are the density $r_k > 0$ and the volume fraction $0 < \alpha_k < 1$ of the heterogeneous component, $k = 1,2$, the common velocity $\boldsymbol{u}$ and absolute temperature $\theta > 0$ of the mixture. They depend on $x = (x_1, \ldots, x_n) \in \Omega$ and $t \geqslant 0$, where Ω is a domain in $\mathbb{R}^n$, $n = 1,2,3$.
Hereafter vector-functions are written in bold,
$\operatorname{div} = \nabla\cdot$, $\nabla = (\partial_1, \ldots, \partial_n)$, $\partial_t = \partial/\partial t$ and $\partial_i = \partial/\partial x_i$.
The symbols $\otimes$ and $\cdot$ correspond to the tensor and scalar products of vectors, the tensor divergence is taken with respect to its first index.

The following additional relations are used

$$\langle \alpha_k \rangle := \alpha_1 + \alpha_2 = 1, \quad \rho = \langle \alpha_k r_k \rangle, \quad \rho\varepsilon = \langle \alpha_k r_k \varepsilon_k(r_k, \theta) \rangle, \tag{4}$$

$$p = p_1(r_1, \theta) = p_2(r_2, \theta) > 0, \tag{5}$$

where $\langle \cdot \rangle$ means **the summation over index** $k = 1, 2$,
$p_k(r_k, \theta)$ and $\varepsilon_k = \varepsilon_k(r_k, \theta)$ are the pressure and specific internal energy of the kth component $(k = 1, 2)$, ρ and ε are the density and specific internal energy of the mixture, and p is the common pressure of the components.
In particular, Eq. (5) means that the pressures p_k of the components are equal to each other, and this is the additional algebraic equation to PDEs (1)-(3) and formula $\langle \alpha_k \rangle = 1$ that is required to define all the sought functions listed above.
We apply **the stiffened gas equations of state** in the well-known form

$$p_k(r_k, \theta) = R_k r_k \theta - p_{*k}, \quad \varepsilon_k(r_k, \theta) = c_{Vk}\theta + \frac{p_{*k}}{r_k} + \varepsilon_{0k}, \tag{6}$$

$R_k > 0$, $c_{Vk} > 0$, $p_{*k} \geqslant 0$ and ε_{0k} are given physical constants, $k = 1, 2$. Also $R_k = (\gamma_k - 1)c_{Vk}$, where $\gamma_k > 1$; let $c_{pk} = \gamma_k c_{Vk}$.
The perfect polytropic gas case corresponds to $p_{*k} = \varepsilon_{0k} = 0$.

The classical Navier-Stokes viscosity tensor and the Fourier heat flux

$$\Pi^{NS} = \mu\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right) + \left(\lambda - \tfrac{2}{3}\mu\right)(\operatorname{div} \boldsymbol{u})\mathbb{I}, \quad -\boldsymbol{q}^F = \varkappa \nabla \theta, \tag{7}$$

where $\mu \geqslant 0$, $\lambda \geqslant 0$ and $\varkappa \geqslant 0$ are the total viscosity and heat conductivity coefficients, $\nabla \boldsymbol{u} = \{\partial_i u_j\}_{i,j=1}^n$ and $\mathbb{I}$ is the n-th order unit tensor. For $\mu = \lambda = 0$ and $\varkappa = 0$, these terms vanish.
Also $\boldsymbol{f}$ is the given density of body forces.
We omit the phase transfer terms here but add the Navier-Stokes ones.

We define the alternative density $\rho_k = \alpha_k r_k$ of the k-th component.
Consider the following relations

$$\rho = \langle \rho_k \rangle, \quad \sigma^{(k)} = \sigma^{(k)}(\rho_1, \rho_2) = \frac{R_k \rho_k}{c_V \rho} > 0, \quad \langle \sigma^{(k)} \rangle = \frac{R}{c_V} = \gamma - 1, \tag{8}$$

$$\rho R = \langle \rho_k R_k \rangle, \quad \rho c_V = \langle \rho_k c_{Vk} \rangle.$$

For p, **the quadratic equation** holds

$$p^2 - bp - c = 0, \tag{9}$$

with the coefficients

$$b = \langle \sigma^{(k)}(\rho(\varepsilon - \varepsilon_0) - p_{*k}) - p_{*k} \rangle, \tag{10}$$

$$\begin{aligned} c &= \sigma^{(1)}(\rho(\varepsilon - \varepsilon_0) - p_{*1})p_{*2} + \sigma^{(2)}(\rho(\varepsilon - \varepsilon_0) - p_{*2})p_{*1} - p_{*1}p_{*2} \\ &= (\sigma^{(1)}p_{*2} + \sigma^{(2)}p_{*1})\rho(\varepsilon - \varepsilon_0) - \gamma p_{*1}p_{*2}. \end{aligned} \tag{11}$$

Let $d := b^2 + 4c$ be its discriminant. For $d > 0$, the quadratic Eq. (9) has the roots

$$p_\pm = p_\pm(\rho_1, \rho_2, \rho\varepsilon) = \tfrac{1}{2}(b \pm \sqrt{d}), \quad p_- < p_+. \tag{12}$$

But for $p_{*1}p_{*2} \neq 0$ (this case arises in some applications), the property $d > 0$ and the correct choice of the physical root are not obvious a priori and are analyzed below.

Proposition 1

Let $\Delta_* := p_{*2} - p_{*1}$. *The following formulas hold*

$$b = p_+ + p_-, \quad c = -p_+ p_- \geqslant 0, \tag{13}$$

where

$$p_+ = \langle \alpha_k p_k \rangle = R\rho\theta - \langle \alpha_k p_{*k} \rangle > 0, \; p_- = -\left(\alpha_1 p_{*2} + \alpha_2 p_{*1} + \frac{\alpha_1 \alpha_2}{c_V \rho \theta} \Delta_*^2 \right) \leqslant 0. \tag{14}$$

Consequently, $d > 0$, *thus, these* $p_\pm$ *and those given by formula* (12) *are the same.*

This Proposition guarantees that p_+ is the physical root and p_- is the parasitic one.
The found formula for p_- is also of interest since it allows to prove additional results.

The balance PDEs for the mass, kinetic and internal energies of the mixture

$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) = 0, \tag{15}$$

$$\tfrac{1}{2}\partial_t(\rho|\boldsymbol{u}|^2) + \tfrac{1}{2}\operatorname{div}\left(\rho|\boldsymbol{u}|^2\boldsymbol{u}\right) + \boldsymbol{u}\cdot\nabla p = (\operatorname{div}\Pi^{NS})\cdot\boldsymbol{u} + \rho\boldsymbol{f}\cdot\boldsymbol{u},$$

$$\partial_t(\rho\varepsilon) + \operatorname{div}(\rho\varepsilon\boldsymbol{u}) + p\operatorname{div}\boldsymbol{u} = \operatorname{div}(-\boldsymbol{q}^F) + \Pi^{NS} : \nabla\boldsymbol{u} + Q \tag{16}$$

are sequentially derived in a standard manner. Here : denotes the scalar product of tensors. In particular, Eq. (15) arises by applying $\langle\cdot\rangle$ to Eqs. (1).

Proposition 2

The squared speed of sound and the balance PDE for p_+ hold

$$c_s^2 := \partial_\rho p_+ + \frac{p_+}{\rho^2}\partial_\varepsilon p_+ = \frac{\gamma(p_+ + p_{*1})(p_+ + p_{*2})}{\rho\sqrt{d}} > 0, \tag{17}$$

$$\partial_t p_+ + \boldsymbol{u}\cdot\nabla p_+ + \rho c_s^2 \operatorname{div}\boldsymbol{u} = \frac{c_s^2}{\gamma c_V \theta}\left(\operatorname{div}(-\boldsymbol{q}^F) + \Pi^{NS} : \nabla\boldsymbol{u} + Q\right), \tag{18}$$

where the derivatives ∂_ρ and ∂_ε are taken in assumption that ε_0 and $\sigma^{(k)}$, $k = 1, 2$, are constant in (10)-(11) following the literature.

The quasi-homogeneous four-equation statement. The balance PDEs for the mass of components, total momentum and total energy

$$\partial_t \rho_k + \operatorname{div}(\rho_k \boldsymbol{u}) = 0, \quad k = 1, 2, \tag{19}$$

$$\partial_t (\rho \boldsymbol{u}) + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = \operatorname{div} \Pi^{NS} + \rho \boldsymbol{f}, \tag{20}$$

$$\partial_t \left(\tfrac{1}{2} \rho |\boldsymbol{u}|^2 + \rho \varepsilon \right) + \operatorname{div} \left((\tfrac{1}{2} \rho |\boldsymbol{u}|^2 + \rho \varepsilon + p) \boldsymbol{u} \right)$$

$$= \operatorname{div}(-\boldsymbol{q}^F + \Pi^{NS} \boldsymbol{u}) + \rho \boldsymbol{u} \cdot \boldsymbol{f}. \tag{21}$$

Here the main sought functions are the alternative densities $\rho_k > 0$, $k = 1, 2$, the velocity $\boldsymbol{u}$ and the specific internal energy ε of the mixture. Also $\rho = \langle \rho_k \rangle$, but formulas (4) for ρ_k and $\rho\varepsilon$ and (6) for p_k are not in use. The pressure p and temperature θ are given by the formulas

$$p(\rho_1, \rho_2, \varepsilon) = p_+ = \frac{1}{2}(b + \sqrt{d}), \quad \theta(\rho_1, \rho_2, \varepsilon) = \frac{\rho(\varepsilon - \varepsilon_0) + p}{\gamma c_V \rho}. \tag{22}$$

Recall that here $d = b^2 + 4c$, with $b = b(\rho_1, \rho_2, \varepsilon)$ and $c = c(\rho_1, \rho_2, \varepsilon)$ given in definitions (10), (11) and (8).

We emphasize that this system does not contain α_k and $r_k = \rho_k/\alpha_k$, $k = 1,2$, although they can be computed a posteriori, we have

$$\alpha_k = \frac{R_k \rho_k \theta}{p_+ + p_{*k}}, \quad k = 1,2. \tag{23}$$

This formula and the property $\langle \alpha_k \rangle = 1$ imply an alternative formula for θ:

$$\theta = \left\langle \frac{R_k \rho_k}{p_+ + p_{*k}} \right\rangle^{-1}, \tag{24}$$

that we apply in our computations below. For computing r_k, the formula $r_k = (p_+ + p_{*k})/(R_k \theta)$ seems to be more reliable.
Importantly, the quasi-homogeneous form **is equivalent** to the original heterogeneous one. In particular, formulas (23) and (24) imply that

$$\langle \alpha_k \rangle = 1, \quad p_k = R_k r_k \theta - p_{*k} = R_k \frac{\rho_k}{\alpha_k} \theta - p_{*k} = p_+, \quad k = 1,2,$$

see the first equation of state (6), and lead to Eqs. (5).

Regularized systems of PDEs for the dynamics of quasi-homogeneous mixtures of stiffened gases

Now we accomplish the formal regularization procedure first suggested in [1] for the single-component gas. In the balance PDEs for the mass of components (19), the total momentum (20) and the total energy of the mixture (21), we accomplish respectively the following changes

$$\rho_k \boldsymbol{u} \ \to \ \rho_k \boldsymbol{u} + \tau \partial_t (\rho_k \boldsymbol{u}),$$

$$\operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p - \rho \boldsymbol{f} \to \ \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u} + \tau \partial_t (\rho \boldsymbol{u} \otimes \boldsymbol{u}))$$

$$+ \nabla (p + \tau \partial_t p) - (\rho + \tau \partial_t \rho) \boldsymbol{f},$$

$$(E + p)\boldsymbol{u} \ \to \ (E + p)\boldsymbol{u} + \tau \partial_t \big((E + p)\boldsymbol{u}\big),$$

$$\rho \boldsymbol{u} \cdot \boldsymbol{f} \ \to \ \big(\rho \boldsymbol{u} + \tau \partial_t (\rho \boldsymbol{u})\big) \cdot \boldsymbol{f},$$

where $E = (1/2)\rho |\boldsymbol{u}|^2 + \rho \varepsilon$ is the total mixture energy and $\tau > 0$ is a regularization parameter which can depend on all the sought functions.

[1] Злотник А. А. О построении квазигазодинамических систем уравнений и баротропной системе с потенциальной массовой силой // Матем. моделир. 2012. Т. 24. № 4. С. 65-79.

These changes lead from the original Navier-Stokes-Fourier-type system (19)-(21) to its following regularized QGD version

$$\partial_t \rho_k + \operatorname{div}\big(\rho_k(\boldsymbol{u}-\boldsymbol{w}_k)\big) = 0, \quad k=1,2,$$

$$\partial_t(\rho\boldsymbol{u}) + \operatorname{div}(\rho(\boldsymbol{u}-\boldsymbol{w})\otimes\boldsymbol{u}) + \nabla p = \operatorname{div}(\Pi^{NS}+\Pi^{\tau}) + \big(\rho - \tau\operatorname{div}(\rho\boldsymbol{u})\big)\boldsymbol{f},$$

$$\partial_t E + \operatorname{div}\big((E+p)(\boldsymbol{u}-\boldsymbol{w})\big) = \operatorname{div}(-\boldsymbol{q}^F - \boldsymbol{q}^{\tau} + (\Pi^{NS}+\Pi^{\tau})\boldsymbol{u}) + \rho(\boldsymbol{u}-\boldsymbol{w})\cdot\boldsymbol{f},$$

the unknown functions are the same. The regularizing velocities

$$\boldsymbol{w}_k := \frac{\tau}{\rho_k}\operatorname{div}(\rho_k\boldsymbol{u})\boldsymbol{u} + \widehat{\boldsymbol{w}}, \quad \widehat{\boldsymbol{w}} = \tau\Big((\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \frac{1}{\rho}\nabla p - \boldsymbol{f}\Big), \tag{25}$$

$$\boldsymbol{w} := \Big\langle \frac{\rho_k}{\rho}\boldsymbol{w}_k \Big\rangle = \frac{\tau}{\rho}\operatorname{div}(\rho\boldsymbol{u}\otimes\boldsymbol{u} + \nabla p - \rho\boldsymbol{f}) = \frac{\tau}{\rho}\operatorname{div}(\rho\boldsymbol{u})\boldsymbol{u} + \widehat{\boldsymbol{w}}, \tag{26}$$

with $k=1,2$, the regularizing viscous stress and heat flux

$$\Pi^{\tau} := \rho\boldsymbol{u}\otimes\widehat{\boldsymbol{w}} + \tau\big(\boldsymbol{u}\cdot\nabla p + \rho c_s^2\operatorname{div}\boldsymbol{u}\big)\mathbb{I}, \tag{27}$$

$$-\boldsymbol{q}^{\tau} := \tau\Big(\boldsymbol{u}\cdot\Big(\rho\nabla\varepsilon - \frac{p}{\rho}\nabla\rho\Big)\Big)\boldsymbol{u}. \tag{28}$$

Finite-difference scheme

Further, we consider the 1D case with $\Omega = (-X, X)$ and define the main and auxiliary uniform meshes

$$\bar{\omega}_h = \{x_i = -X + ih;\, 0 \leqslant i \leqslant N\},$$

$$\omega_h^* = \{x_{i+1/2} = -X + (i+0.5)h;\, 0 \leqslant i \leqslant N-1\},$$

on $[-X, X]$, with the step $h = 2X/N$. Let $\omega_h = \bar{\omega}_h \backslash \{-X, X\}$.
We also define the nonuniform mesh $\bar{\omega}^{\Delta t} = \{t_0 = 0 < t_1 < \ldots < t_{\overline{m}} = t_{fin}\}$ in time, with the steps $\Delta t_m = t_{m+1} - t_m$. Let $\check{\omega}^{\Delta t} = \bar{\omega}^{\Delta t} \backslash \{t_{fin}\}$.
Denote by $H(\omega)$ the space of functions given on a mesh ω.
For $v \in H(\bar{\omega}_h)$, $w \in H(\omega_h^*)$ and $y \in H(\bar{\omega}^\tau)$, we introduce the averages and difference quotients

$$[v]_{i+1/2} = 0.5(v_i + v_{i+1}), \quad v_{i+1/2} = \frac{v_{i+1} - v_i}{h},$$

$$[w]_i^* = 0.5(w_{i-1/2} + w_{i+1/2}), \quad \delta^* w_i = \frac{w_{i+1/2} - w_{i-1/2}}{h}, \quad \delta_t y^m = \frac{y^{m+1} - y^m}{\Delta t_m},$$

where $v_i = v(x_i)$, $w_{i+1/2} = w(x_{i+1/2})$ and $y^m = y(t_m)$.

For the regularized QGD balance PDEs in the 1D case, we construct the explicit two-level in time and **symmetric three-point in space** discrete balance Eqs. **without limiters** for the mass of the components and the momentum and total energy of the gas mixture

$$\delta_t \rho_k + \delta^* \big([\rho_k]([u] - w_k) \big) = 0, \quad k = 1,2, \tag{29}$$

$$\delta_t (\rho u) + \delta^* \big([\rho]([u] - w)[u] + [p] \big) = \delta^* \Pi, \tag{30}$$

$$\begin{aligned} \delta_t \big(\tfrac{1}{2} \rho u^2 + \rho \varepsilon \big) + \delta^* \big\{ \big(\tfrac{1}{2} [\rho] u_- u_+ + [\rho \varepsilon] + [p] \big) ([u] - w) - \tfrac{1}{4} h^2 (\delta p) \delta u \big\} \\ = \delta^* (-q + \Pi [u]) + [Q]^* \end{aligned} \tag{31}$$

on $\omega_h \times \breve{\omega}^{\Delta t}$. The main sought functions $\rho_1 > 0, \rho_2 > 0$, u, ε (in fact, $\rho\varepsilon$), and the functions p and θ, are defined on the main mesh $\bar{\omega}_h \times \bar{\omega}^{\Delta t}$.
Also p and θ (in the q-term) are given by the first formula (22) and formula (24), with $d = b^2 + 4c$ and their coefficients defined by (8), (10) and (11).
In Eq. (31), the nonstandard term $u_- u_+$ (like the geometric mean for u^2) instead of $[u^2]$ or $[u]^2$ and the additional term $-(1/4)h^2(\delta p)\delta u$ allows us to ensure a more natural form of the important discrete balance equation for the mixture internal energy without the spatial mesh imbalances.

We discretize the regularizing velocities (25)-(26) in the form

$$w_k = \frac{[\tau]}{[\rho_k]}\,[u]\delta(\rho_k u) + \widehat{w}, \quad \widehat{w} = \frac{[\tau]}{[\rho]}\,([\rho][u]\delta u + \delta p), \tag{32}$$

$$w = \left\langle \frac{[\rho_k]}{[\rho]} w_k \right\rangle = \frac{[\tau]}{[\rho]}\,[u]\delta(\rho u) + \widehat{w} \tag{33}$$

with $k = 1, 2$ and the viscous stress and heat flux as follows

$$\Pi = \nu\delta u + [u][\rho]\widehat{w} + [\tau]\big([u]\delta p + [\rho c_s^2]\delta u\big),$$

$$-q = \varkappa\delta\theta + [\tau]\left\{\left(\delta(\rho\varepsilon) - \frac{[\rho\varepsilon] + [p]}{[\rho]}\delta\rho\right)[u]^2\right\}.$$

Here the squared speed of sound c_s^2 is given by the second formula in (17), and c_V and γ are introduced above.
The functions w_k, $\widehat{w}$, w, Π, $\nu = (4/3)\mu + \lambda$, q and $\varkappa$ are defined on the auxiliary mesh $\omega_h^* \times \bar{\omega}^{\Delta t}$, but τ is defined on $\bar{\omega}_h \times \bar{\omega}^{\Delta t}$.

We take τ, ν and $\varkappa$ in the form

$$\tau = \frac{ah}{c_s}, \quad \nu = a_S[\tau][p], \quad \varkappa = a_{Pr}[\tau][c_p][p]$$

that is formally analogous to the single-component gas case.
So τ is h-dependent, ν and $\varkappa$ are artificial viscosity coefficients, with the parameter $a > 0$, the Schmidt and inverse Prandtl numbers for the mixture $a_S \geqslant 0$ and $a_{Pr} > 0$ used as adjusting numerical parameters.
For the QGD regularization, in many tests, $a_S = 0$ is possible.

The initial data $(\rho_1, \rho_2, u, \rho\varepsilon) = (\rho_1^0, \rho_2^0, u^0, (\rho\varepsilon)^0)$ (or equivalent ones) are given on $\bar{\omega}_h$.
Below the time steps are chosen automatically according to the formulas

$$\Delta t_m = \frac{\beta h}{\max_i(c_{si}^m + |u_i^m|)}, \ 0 \leqslant m < \overline{m} - 1,$$

$$\Delta t_{\overline{m}-1} = t_{fin} - t_{\overline{m}-1} \leqslant \frac{\beta h}{\max_i(c_{si}^{\overline{m}-1} + |u_i^{\overline{m}-1}|)},$$

where $\beta > 0$ is the Courant-type parameter.

Test B. Water-to-air shock tube problem

Таблица 1: Stiffened gas parameters

Substance	γ	c_V, J/(kg K)	p_*, Pa	ε_0, J/kg
Air	1.4	720	0	0
Water	2.8	1495	$8.5 \cdot 10^8$	0

In this test [2], we have a 10 m long tube separated into two halves, both of which contain a mixture of air and water but in different proportions, with

$$(p_0, u_0, \theta_0) = \begin{cases} (2 \cdot 10^7 \text{ Pa}, 0 \text{ m/s}, 308.15 \text{ K}), & -5 \leqslant x < 0 \\ (10^7 \text{ Pa}, 0 \text{ m/s}, 308.15 \text{ K}), & 0 < x \leqslant 5 \end{cases},$$

and we have $\alpha_1 = 0.25$ in the left half and $\alpha_1 = 0.75$ in the right half. Notice that $a = 2$ is taken, thus, $a > 1$.

[2] Q. Li, S. Fu, A gas-kinetic BGK scheme for gas-water flow, *Comput. Math. Appl.* **61**, 3639–3652 (2011).

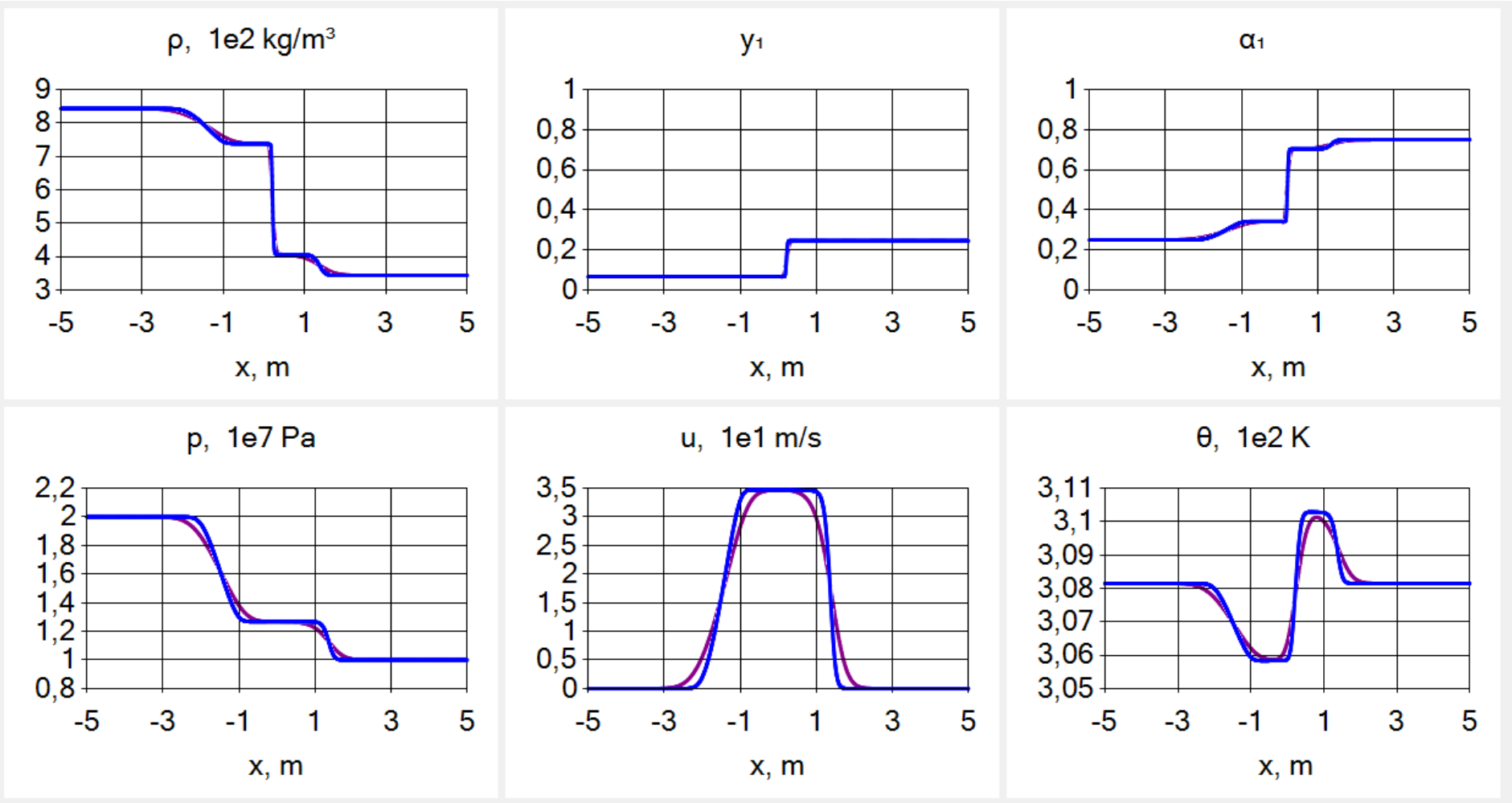

Рис. 1: Numerical results for water-to-air shock tube (test B) for $N = 500$ (dark magenta), 2500 (blue), $a = 2$ and $\beta = 0.1$, $t_{fin} = 6$ ms (the QGD regularization)